Random melting skew Young diagram

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Integrable Structures in Random Matrix Theory and Beyond
MSRI, 10/19/2021

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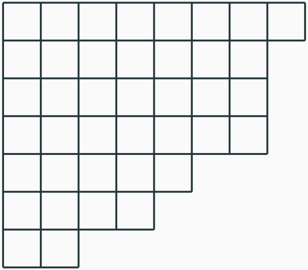
1. Introduction

Introduction



A Young diagram Y_λ, with

$$\lambda = (8, 7, 7, 7, 5, 4, 2).$$

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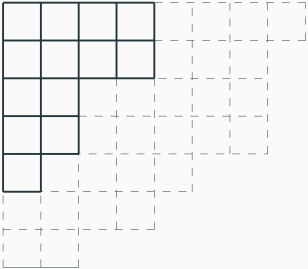
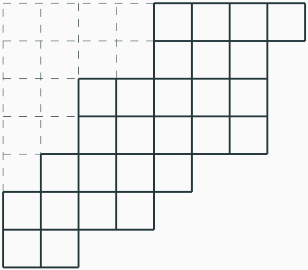


A Young diagram Y_μ, with

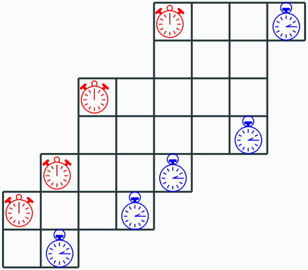
$$\mu = (4, 4, 2, 2, 1).$$



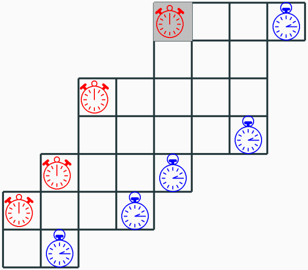
A skew Young diagram $Y_{\lambda/\mu}$, with

$$\lambda = (8, 7, 7, 7, 5, 4, 2), \quad \mu = (4, 4, 2, 2, 1).$$

Introduction



We assign each NW corner a clock  which rings after an independent waiting time with parameter γ_1, and each SE corner a clock  which rings after an independent waiting time with parameter γ_2.



If any clock rings, the associated box melts.

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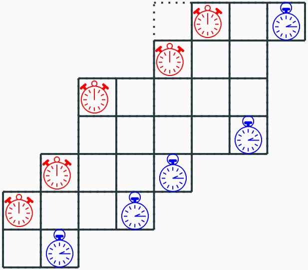

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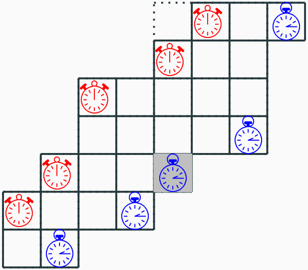

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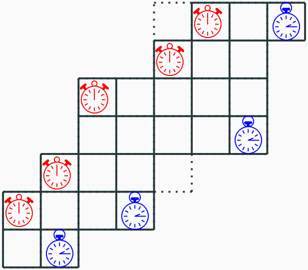

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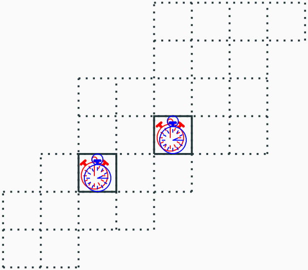
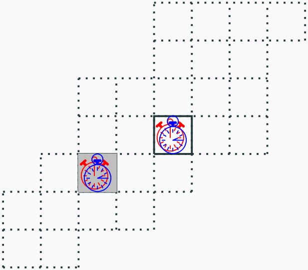


It might be possible that one box has two clocks, or the shape becomes disconnected.

Introduction



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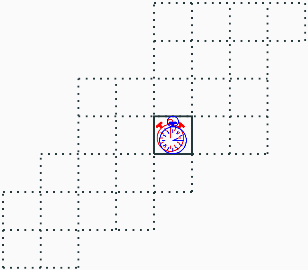

Question: Where is the last melted box and when does it melt?

2. Corner growth model

Corner growth model

Why are we interested in the random melting skew Young diagram?

Corner growth model

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Reason 1: It is a natural generalization of the corner growth model ($\gamma_1 = 0$ or $\gamma_2 = 0$).

Corner growth model: imagine that you burn a paper from a corner.

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Random melting skew Young diagram: imagine that you burn a paper from two corners.

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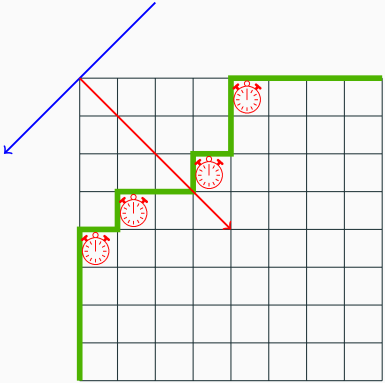
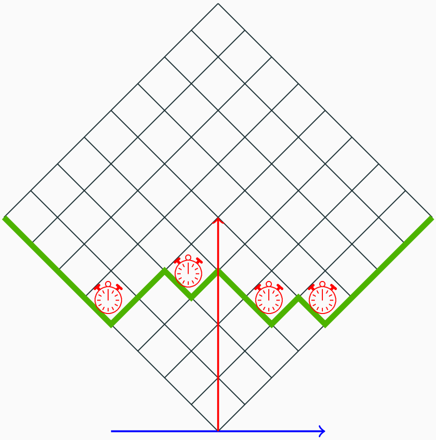


blue arrow: spatial direction

red arrow: temporal direction

green line: height function $h(x, t)$

Corner growth model



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The limiting fluctuations of $h(x, t)$ depend on the initial condition $h(x, 0)$.

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$$h(x, 0) = \begin{cases} 1, & x = 2n + 1, \quad n \in \mathbb{Z}, \\ |x - 2n|, & 2n - 1 < x < 2n + 1, \quad n \in \mathbb{Z}. \end{cases}$$

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Stationary IC, Step-flat IC, Step-Stationary IC, Flat-Stationary IC, etc.

Corner growth model

For step IC, it is known [Johansson00, Johansson03]

$$\frac{h(\mathrm{x}t^{2/3}, t) - c_1 t}{c_2 t^{1/3}} \to \mathcal{A}_2(\mathrm{x}) - \mathrm{x}^2$$

as $t \to \infty$, where $\mathcal{A}_2$ is the Airy$_2$ process [Prähofer-Spohn02].

Corner growth model

For flat IC, [Baik-Rains01], [Borodin,Ferrari,Prähofer,Sasamoto,05-08]

$$\frac{h(\mathrm{x}t^{2/3}, t) - c_1 t}{c_2 t^{1/3}} \to \mathcal{A}_1(\mathrm{x})$$

as $t \to \infty$, where $\mathcal{A}_1$ is the Airy$_1$ process.

Corner growth model

- Other classic initial conditions: $\mathcal{A}_{\mathrm{stat}}(x)$ [Baik-Ferrari-Péché10], $\mathcal{A}_{1\to 2}$ [Borodin-Ferrari-Sasamoto08], $\mathcal{A}_{2\to \mathrm{stat}}$ [Corwin-Ferrari-Péché10], $\mathcal{A}_{1\to \mathrm{stat}}$ [Borodin-Ferrari-Sasamoto09].

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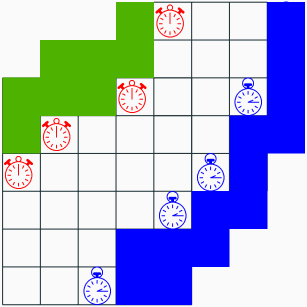
- General initial conditions: [Matetski-Quastel-Remenik16].
- Two-time or more generally multi-time/multi-point in the space-time plane: [Johansson17, Johansson-Rahman19, Liu19].
- More generally, there is a limiting four-parameter random field, the so-called the directed landscape/Airy sheet constructed recently by [Dauvergne-Ortmann-Virag18].

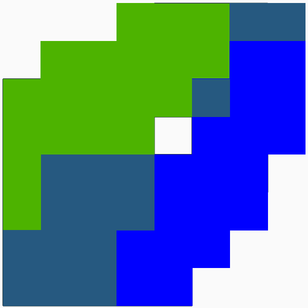
Corner growth model

The random melting skew Young diagram can be viewed two independent corner growth models growing towards each other.



Corner growth model

The last melting box corresponds to the last box which is not covered by green or blue.



Heuristically, when time becomes infinity, the location of the last melting box is related to the argmax of the sum of two Airy-type processes.

3. Geodesic in the directed last passage percolation

Geodesic in DLPP

The corner growth model is equivalent to a directed last passage percolation model.

$w_{i,j} \sim \exp(1)$, i.i.d.

Last passage time

$$L_{\mathbf{p}}(\mathbf{q}) := \max_{\pi:\mathbf{p}\to\mathbf{q}} \sum_{\mathbf{r}\in\pi} w_{\mathbf{r}}$$

More generally,

$$L_{\mathbf{S}}(\mathbf{E}) := \max_{\pi:\mathbf{p}\in\mathbf{S}\to\mathbf{q}\in\mathbf{E}} \sum_{\mathbf{r}\in\pi} w_{\mathbf{r}}.$$

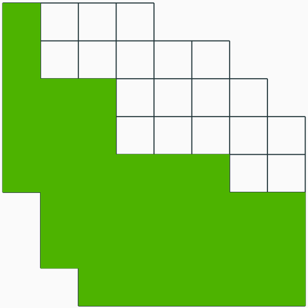
The maximizer is called the *geodesic*. We denote it by $\mathcal{G}_{\mathbf{p}}(\mathbf{q})$ (or $\mathcal{G}_{\mathbf{S}}(\mathbf{E})$).

Geodesic in DLPP

A box (m, n) is on the geodesic $\mathcal{G}_{\mathbf{S}}(\mathbf{E})$ if and only if (m, n) is the only box which does not belong to

$$\{\mathbf{r} : \mathcal{L}_{\mathbf{S}}(\mathbf{r}) \leq t\} \cup \{\mathbf{r} : \mathcal{L}_{\mathbf{r}}(\mathbf{E}) \leq s\}$$

for some $t, s \geq 0$.



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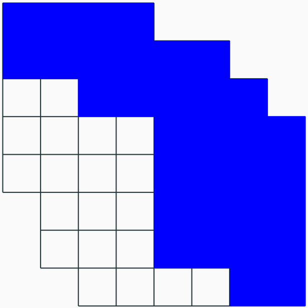

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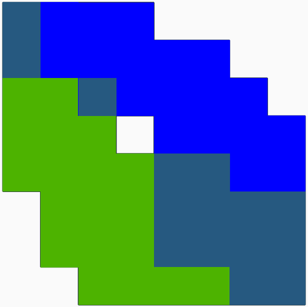

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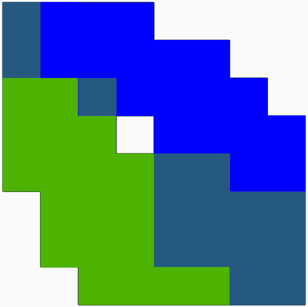

Heuristically the limiting location of the geodesic location is related to the argmax of two independent Airy-type processes.

Geodesic in DLPP

The distribution of the location of the point-to-point geodesic $\mathcal{G}_{\mathbf{p}}(\mathbf{q})$ was only obtained very recently [Liu21]. However, the approach in [Liu21] seems not applicable for the geodesic $\mathcal{G}_{\mathbf{S}}(\mathbf{E})$.

4. Main results

Notations

we use lattice points $(x, y) \in \mathbb{Z}^2_{\geq 1}$ to represent the boxes in the skew Young diagrams, here we let the y-axis goes towards to the south for convenience.

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We denote $Y(t)$ the random melting skew Young diagram with initial condition

$$Y(0) = Y_{\lambda/\mu} = \{(i, j) : \mu_j < i \leq \lambda_j\}, \quad \lambda = (\lambda_1, \lambda_2, \cdots), \quad \mu = (\mu_1, \mu_2, \cdots).$$

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Definition
Define t_{melt} to be the smallest time t such that $Y(t) = \emptyset$. In other words,

$$t_{\mathrm{melt}} := \min_{t \geq 0}\{t : Y(t) = \emptyset\}.$$

We call t_{melt} the melting time. We also define Y_{melt} the last melted box, i.e.,

$$Y_{\mathrm{melt}} := \lim_{\epsilon \downarrow 0} Y(t_{\mathrm{melt}} - \epsilon).$$

Finite time formula

Theorem (Liu21+)

We have

$$\mathbb{P}_{\lambda/\mu}(t_{\text{melt}} \in (t, t+\epsilon), Y_{\text{melt}} = (n, m)) \\ = (\gamma_1 + \gamma_2) \int_t^{t+\epsilon} p_{\lambda/\mu}(\gamma_1 t, \gamma_2 t, m, n) \mathrm{d}t,$$

where the function $p_{\lambda/\mu}(t_1, t_2, m, n)$ is defined in later slides.

Kernels K_μ and $\tilde{K}_\lambda^{(N)}$

The information of $Y_{\lambda/\mu}$ is encoded in two kernels.

Suppose $\mu = (\mu_1, \cdots)$ is a partition. Let $c_{\mu,\kappa}$ be the unique coefficient satisfying the following symmetric function expansion

$$\mathcal{G}_\mu(w_1, \cdots, w_N) := \frac{\det\left[w_i^{-j}(w_i+1)^{\mu_j}\right]_{i,j=1}^N}{\det\left[w_i^{-j}\right]_{i,j=1}^N} = \sum_\kappa c_{\mu,\kappa} p_\kappa(w_1, \cdots, w_N)$$

where N is an arbitrary positive integer satisfying $N > \sum_i \mu_i$, and the summation on the right hand side is running over all possible partitions $\kappa = (\kappa_1, \cdots)$, and the function $p_\kappa(w_1, \cdots, w_N) = \prod_{j:\kappa_j>0} \sum_{i=1}^N w_i^{\kappa_j}$ is the power sum symmetric function.

The function $\mathcal{G}_\mu(w_1, \cdots, w_N)$ is related to the inhomoegeneous Schur polynomial defined by Borodin in [Bor17], and the dual Grothendieck polynomial [Motegi-Sakai13].

Kernels K_μ and $\hat{K}_\lambda^{(N)}$

Define

$$\chi_\mu(v, u) = 1 + \sum_{\kappa \neq (0)} c_{\mu,\kappa} \prod_{j:\kappa_j > 0} (u^{\kappa_j} - v^{\kappa_j}).$$

It is easy to show that

$$\chi_\mu(v, u) = \mathcal{G}_\mu(u, v\xi, v\xi^2, \cdots, v\xi^{N-1})$$

where $\xi = e^{2\pi \mathrm{i}/N}$ is the N-th root of unity provided $N > |\mu|$.

Kernels K_μ and $\hat{K}_\lambda^{(N)}$

Definition

Suppose $\mu = (\mu_1, \cdots)$ *is a partition and* u, v *are two complex numbers. We define the kernel*

$$K_\mu(v,u) = \frac{\chi_\mu(v,u)}{v-u}$$

Suppose λ *is a partition and* N *is an integer satisfying* $\lambda_j = 0$ *for all* $j > N$. *We also define*

$$\hat{K}_\lambda^{(N)}(u,v) = -K_{(\lambda_1-\lambda_N, \lambda_1-\lambda_{N-1}, \cdots, \lambda_1-\lambda_2, 0)}(v,u) \cdot \frac{u^{N+1}(v+1)^{\lambda_1}}{v^{N+1}(u+1)^{\lambda_1}}.$$

K_μ does not depend on N, but $\hat{K}_\lambda^{(N)}$ depends on the parameter N.

Definition of $p_{\lambda/\mu}$

Suppose λ and μ are two partitions satisfying $\lambda_i \geq \mu_i \geq 0$ for all i, and $(m, n) \in Y^{\lambda/\mu}$. We define

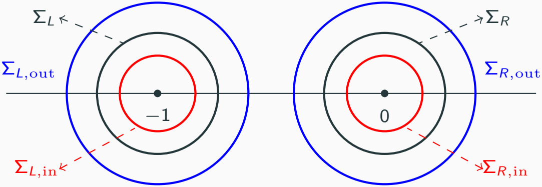
$$p_{\lambda,\mu}(t_1, t_2, m, n) = \oint_0 \frac{\mathrm{dz}}{2\pi \mathrm{i}(1-\mathrm{z})^2} \sum_{k_1,k_2 \geq 1} \frac{1}{(k_1!k_2!)^2} T_{k_1,k_2}(\mathrm{z}; t_1, t_2; m, n),$$

Definition of $p_{\lambda/\mu}$

where

$$
\begin{aligned}
&T_{k_1,k_2}(\mathrm{z}; t_1, t_2; m, n) \\
&:= \prod_{i_1=1}^{k_1}\left(\frac{-\mathrm{z}}{1-\mathrm{z}}\int_{\Sigma_{\mathrm{L,out}}}\frac{\mathrm{d}u_{i_1}^{(1)}}{2\pi\mathrm{i}} + \frac{1}{1-\mathrm{z}}\int_{\Sigma_{\mathrm{L,in}}}\frac{\mathrm{d}u_{i_1}^{(1)}}{2\pi\mathrm{i}}\right)\prod_{i_2=1}^{k_2}\int_{\Sigma_{\mathrm{L}}}\frac{\mathrm{d}u_{i_2}^{(2)}}{2\pi\mathrm{i}} \\
&\quad \prod_{i_1=1}^{k_1}\left(\frac{-\mathrm{z}}{1-\mathrm{z}}\int_{\Sigma_{\mathrm{R,out}}}\frac{\mathrm{d}v_{i_1}^{(1)}}{2\pi\mathrm{i}} + \frac{1}{1-\mathrm{z}}\int_{\Sigma_{\mathrm{R,in}}}\frac{\mathrm{d}v_{i_1}^{(1)}}{2\pi\mathrm{i}}\right)\prod_{i_2=1}^{k_2}\int_{\Sigma_{\mathrm{R}}}\frac{\mathrm{d}v_{i_2}^{(2)}}{2\pi\mathrm{i}} \\
&\quad (\mathrm{z}^{-1}-1)^{k_1}(-1+\mathrm{z})^{k_2}\frac{f_1(U^{(1)})f_2(U^{(2)})}{f_1(V^{(1)})f_2(V^{(2)})}\cdot H(U^{(1)}, U^{(2)}; V^{(1)}, V^{(2)}) \\
&\quad \frac{\Delta(U^{(1)})\Delta(U^{(2)})\Delta(V^{(1)})\Delta(V^{(2)})\Delta(U^{(1)}; V^{(2)})\Delta(V^{(1)}; U^{(2)})}{\Delta(U^{(1)}; V^{(1)})\Delta(U^{(2)}; V^{(2)})\Delta(U^{(1)}; U^{(2)})\Delta(V^{(1)}; V^{(2)})} \\
&\quad \cdot(-1)^{\binom{k_1}{2}+\binom{k_2}{2}}\det\left[K_\mu(v_{j_1}^{(1)}, u_{i_1}^{(1)})\right]_{i_1,j_1=1}^{k_1}\det\left[\hat{K}_\lambda^{(N)}(u_{j_2}^{(2)}, v_{i_2}^{(2)})\right]_{i_2,j_2=1}^{k_2}.
\end{aligned}
$$

Definition of $p_{\lambda/\mu}$



The contours

Definition of $p_{\lambda/\mu}$

The functions f_1 and f_2 are defined by

$$f_1(w) = f_1(w; t_1, m, n) = e^{t_1 w} w^n (w+1)^{-m},$$
$$f_2(w) = f_2(w; t_2, m, n) := e^{t_2 w} w^{-n} (w+1)^{m-1}.$$

The function H is defined by

$$\begin{aligned}
&H(U^{(1)}, U^{(2)}; V^{(1)}, V^{(2)}) \\
&:= \prod_{i_1=1}^{k_1} \frac{v_{i_1}^{(1)}(u_{i_1}^{(1)}+1)}{u_{i_1}^{(1)}(v_{i_1}^{(1)}+1)} \left(\sum_{i_1=1}^{k_1} (u_{i_1}^{(1)} - v_{i_1}^{(1)}) - \sum_{i_2=1}^{k_2} (u_{i_2}^{(2)} - v_{i_2}^{(2)}) - 1 \right. \\
&\quad \left. + \prod_{i_1=1}^{k_1} \frac{v_{i_1}^{(1)}+1}{u_{i_1}^{(1)}+1} \prod_{i_2=1}^{k_2} \frac{u_{i_2}^{(2)}+1}{v_{i_2}^{(2)}+1} \right) \\
&- \prod_{i_2=1}^{k_2} \frac{v_{i_2}^{(2)}(u_{i_2}^{(2)}+1)}{u_{i_2}^{(2)}(v_{i_2}^{(2)}+1)} \left(\sum_{i_1=1}^{k_1} (u_{i_1}^{(1)} - v_{i_1}^{(1)}) - \sum_{i_2=1}^{k_2} (u_{i_2}^{(2)} - v_{i_2}^{(2)}) + 1 \right. \\
&\quad \left. - \prod_{i_1=1}^{k_1} \frac{u_{i_1}^{(1)}+1}{v_{i_1}^{(1)}+1} \prod_{i_2=1}^{k_2} \frac{v_{i_2}^{(2)}+1}{u_{i_2}^{(2)}+1} \right).
\end{aligned}$$

Remarks

1. Although $\hat{K}_{\lambda}^{(N)}$ has the parameter N, the function $p_{\lambda/\mu}$ does not depend on N.
2. We are able to numerically verify the following cases.
   (1) $\lambda = (2)$ and $\mu = (0)$, $m = n = 1$. $p(s,t) = te^{-s-t}$.
   (2) $\lambda = (2,2), \mu = (0)$, $m = n = 1$. $p(s,t) = e^{-s-t}(t^2 - 2t + 2 - 2e^{-t})$.
   (3) $\lambda = (2,2), \mu = (0)$, $m = 1, n = 2$.
   $p(s,t) = e^{-s-t}(st - 1 + e^{-s} + e^{-t} - e^{-s-t})$.
   (4) $\lambda = (3)$ and $\mu = (0)$, $m = n = 1$. $p(s,t) = \frac{1}{2}t^2 e^{-s-t}$.
   (5) $\lambda = (3)$ and $\mu = (0)$, $m = 2, n = 1$. $p(s,t) = ste^{-s-t}$.
   (6) $\lambda = (2,1)$, $\mu = (0)$, $m = n = 1$. $p(s,t) = e^{-s-t}(2t - 2 + 2e^{-t})$.

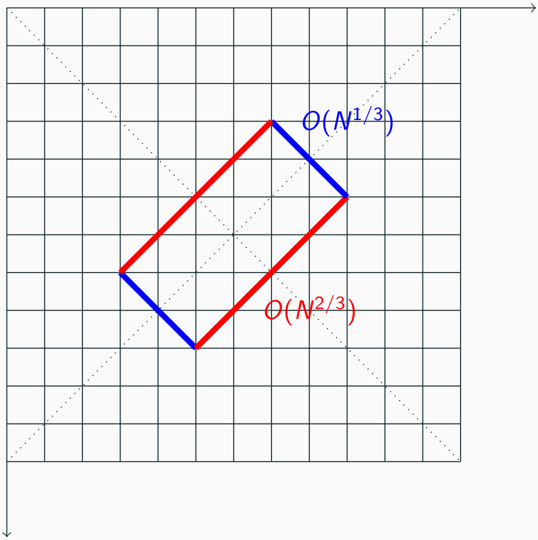
Theorem (Step case)
When $\lambda = (\underbrace{M, \cdots, M}_{N})$ *and* $\mu = (0)$*, with the following scaling*

$$
\begin{aligned}
M &= \alpha N, \\
t &= (1+\sqrt{\alpha})^2 N + \mathrm{t}\cdot\alpha^{-1/6}(1+\sqrt{\alpha})^{4/3} N^{1/3}, \\
m &= \frac{\gamma_1}{\gamma_1+\gamma_2}\alpha N - \mathrm{x}\cdot\alpha^{2/3}(1+\sqrt{\alpha})^{-1/3}N^{2/3} - \mathrm{y}\cdot\alpha^{5/6}(1+\sqrt{\alpha})^{-2/3}N^{1/3}, \\
n &= \frac{\gamma_1}{\gamma_1+\gamma_2} N + \mathrm{x}\cdot\alpha^{1/6}(1+\sqrt{\alpha})^{-1/3}N^{2/3} - \mathrm{y}\cdot\alpha^{-1/6}(1+\sqrt{\alpha})^{-2/3}N^{1/3}
\end{aligned}
$$

then

$$
\begin{aligned}
&\lim_{N\to\infty} \alpha^{1/3}(1+\sqrt{\alpha})^{4/3} N^{4/3} p_{\lambda/\mu}\left(\frac{\gamma_1}{\gamma_1+\gamma_2}t, \frac{\gamma_2}{\gamma_1+\gamma_2}t, m, n\right) \\
&= \mathrm{p}_{\mathrm{s/s}}\left(\mathrm{t}, \mathrm{x}, \mathrm{y}; \frac{\gamma_1}{\gamma_1+\gamma_2}\right).
\end{aligned}
$$

Limit theorems

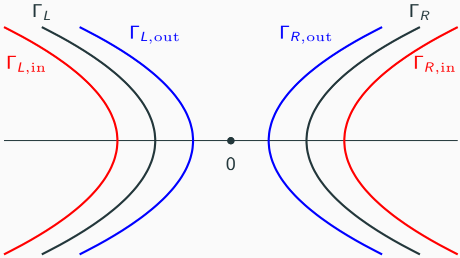


The function $\mathrm{p}_{\mathrm{s/s}}$

$$\mathrm{p}_{\mathrm{s/s}}(\mathrm{t},\mathrm{x},\mathrm{y};\gamma) := \oint_0 \frac{\mathrm{d}\mathrm{z}}{2\pi\mathrm{i}(1-\mathrm{z})^2} \sum_{k_1,k_2\geq 1} \frac{1}{(k_1!k_2!)^2} \mathrm{T}_{k_1,k_2}(\mathrm{z};\mathrm{t},\mathrm{x},\mathrm{y};\gamma)$$

$$\begin{aligned}
&\mathrm{T}_{k_1,k_2}(\mathrm{z};\mathrm{t},\mathrm{x},\mathrm{y};\gamma) \\
&:= \prod_{i_1=1}^{k_1}\left(\frac{1}{1-\mathrm{z}}\int_{\Gamma_{\mathrm{L,in}}}\frac{\mathrm{d}\xi_{i_1}^{(1)}}{2\pi\mathrm{i}} - \frac{\mathrm{z}}{1-\mathrm{z}}\int_{\Gamma_{\mathrm{L,out}}}\frac{\mathrm{d}\xi_{i_1}^{(1)}}{2\pi\mathrm{i}}\right)\prod_{i_2=1}^{k_2}\int_{\Gamma_{\mathrm{L}}}\frac{\mathrm{d}\xi_{i_2}^{(2)}}{2\pi\mathrm{i}} \\
&\quad \prod_{i_1=1}^{k_1}\left(\frac{1}{1-\mathrm{z}}\int_{\Gamma_{\mathrm{R,in}}}\frac{\mathrm{d}\eta_{i_1}^{(1)}}{2\pi\mathrm{i}} - \frac{\mathrm{z}}{1-\mathrm{z}}\int_{\Gamma_{\mathrm{R,out}}}\frac{\mathrm{d}\eta_{i_1}^{(1)}}{2\pi\mathrm{i}}\right)\prod_{i_2=1}^{k_2}\int_{\Gamma_{\mathrm{R}}}\frac{\mathrm{d}\eta_{i_2}^{(2)}}{2\pi\mathrm{i}} \\
&\quad (1-\mathrm{z})^{k_2}\left(1-\frac{1}{\mathrm{z}}\right)^{k_1}\cdot\frac{\mathrm{f}_1(\boldsymbol{\xi}^{(1)};\gamma)\mathrm{f}_2(\boldsymbol{\xi}^{(2)};1-\gamma)}{\mathrm{f}_1(\boldsymbol{\eta}^{(1)};\gamma)\mathrm{f}_2(\boldsymbol{\eta}^{(2)};1-\gamma}\cdot\mathrm{H}(\boldsymbol{\xi}^{(1)},\boldsymbol{\eta}^{(1)};\boldsymbol{\xi}^{(2)},\boldsymbol{\eta}^{(2)}) \\
&\quad \cdot\prod_{\ell=1}^{2}\frac{\left(\Delta(\boldsymbol{\xi}^{(\ell)})\right)^2\left(\Delta(\boldsymbol{\eta}^{(\ell)})\right)^2}{\left(\Delta(\boldsymbol{\xi}^{(\ell)};\boldsymbol{\eta}^{(\ell)})\right)^2}\cdot\frac{\Delta(\boldsymbol{\xi}^{(1)};\boldsymbol{\eta}^{(2)})\Delta(\boldsymbol{\eta}^{(1)};\boldsymbol{\xi}^{(2)})}{\Delta(\boldsymbol{\xi}^{(1)};\boldsymbol{\xi}^{(2)})\Delta(\boldsymbol{\eta}^{(1)};\boldsymbol{\eta}^{(2)})}
\end{aligned}$$

The function $p_{s/s}$



The contours

The function $\mathrm{p}_{\mathrm{s/s}}$

The vectors $\boldsymbol{\xi}^{(\ell)} = (\xi_1^{(\ell)}, \cdots, \xi_{i_\ell}^{(\ell)})$ and $\boldsymbol{\eta}^{(\ell)} = (\eta_1^{(\ell)}, \cdots, \eta_{i_\ell}^{(\ell)})$ for $\ell \in \{1, 2\}$, the functions $\mathrm{f}_1, \mathrm{f}_2$ are defined by

$$\mathrm{f}_1(\zeta; \gamma) := \exp\left(-\frac{\gamma}{3}\zeta^3 - \frac{1}{2}\mathrm{x}\zeta^2 + (\mathrm{y} + \gamma \mathrm{t})\,\zeta\right),$$
$$\mathrm{f}_2(\zeta; \gamma) := \exp\left(-\frac{\gamma}{3}\zeta^3 + \frac{1}{2}\mathrm{x}\zeta^2 + (-\mathrm{y} + \gamma \mathrm{t})\,\zeta\right),$$

and the function H is defined by

$$\mathrm{H}(\boldsymbol{\xi}^{(1)}, \boldsymbol{\eta}^{(1)}; \boldsymbol{\xi}^{(2)}, \boldsymbol{\eta}^{(2)}) = \frac{1}{12}\mathrm{S}_1^4 + \frac{1}{4}\mathrm{S}_2^2 - \frac{1}{3}\mathrm{S}_1\mathrm{S}_3$$

with

$$\begin{aligned} \mathrm{S}_\ell &= \mathrm{S}_\ell(\boldsymbol{\xi}^{(1)}, \boldsymbol{\eta}^{(1)}; \boldsymbol{\xi}^{(2)}, \boldsymbol{\eta}^{(2)}) \\ &= \sum_{i_1=1}^{k_1}\left(\left(\xi_{i_1}^{(1)}\right)^\ell - \left(\eta_{i_1}^{(1)}\right)^\ell\right) - \sum_{i_2=1}^{k_2}\left(\left(\xi_{i_2}^{(2)}\right)^\ell - \left(\eta_{i_2}^{(2)}\right)^\ell\right). \end{aligned}$$

The function $\mathrm{p}_{\mathrm{s/s}}$

Corollary
Let $\mathcal{A}_2^{(1)}$ *and* $\mathcal{A}_2^{(2)}$ *be two independent Airy*$_2$ *processes. Denote* $\hat{\mathcal{A}}_2^{(\ell)}(x) = \mathcal{A}_2^{(\ell)}(x) - x^2$, $\ell = 1, 2$. *Denote*

$$\mathcal{T} = \mathrm{argmax}_{\mathrm{x}} \left(\gamma^{1/3} \hat{\mathcal{A}}_2^{(1)} \left(\frac{\mathrm{x}}{2\gamma^{2/3}} \right) + (1-\gamma)^{1/3} \hat{\mathcal{A}}_2^{(2)} \left(\frac{\mathrm{x}}{2(1-\gamma)^{2/3}} \right) \right).$$

Then $\mathrm{p}_{\mathrm{s/s}}\left(\mathrm{h}_1 + \mathrm{h}_2, \mathrm{x}, (1-\gamma)\mathrm{h}_1 - \gamma \mathrm{h}_2; \gamma\right)$ *is the joint probability density function at*

$$\left(\gamma^{1/3} \hat{\mathcal{A}}_2^{(1)} \left(\frac{\mathcal{T}}{2\gamma^{2/3}} \right), (1-\gamma)^{1/3} \hat{\mathcal{A}}_2^{(2)} \left(\frac{\mathcal{T}}{2(1-\gamma)^{2/3}} \right), \mathcal{T} \right) = (\mathrm{h}_1, \mathrm{h}_2, \mathrm{x}).$$

This is consistent with the known result in [Liu21].

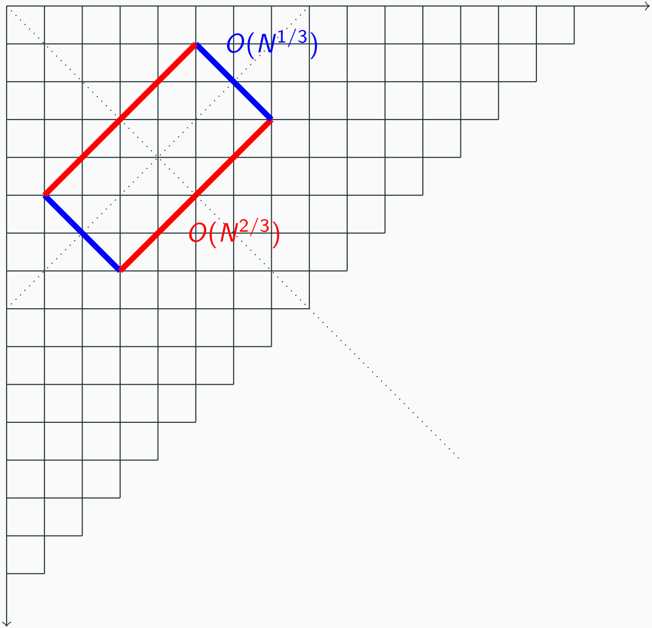
Theorem (Flat case)
When $\lambda = (N, N-1, \cdots, 1)$ *and* $\mu = (0)$, *under the following scaling*

$$\begin{aligned} t &= 2N + \mathrm{t} \cdot 2^{1/3} N^{1/3}, \\ m &= \frac{\gamma_1}{2(\gamma_1 + \gamma_2)} N - \mathrm{x} \cdot 2^{-4/3} N^{2/3} - \mathrm{y} \cdot 2^{-5/3} N^{1/3}, \\ n &= \frac{\gamma_1}{2(\gamma_1 + \gamma_2)} N + \mathrm{x} \cdot 2^{-4/3} N^{2/3} - \mathrm{y} \cdot 2^{-5/3} N^{1/3} \end{aligned}$$

then

$$\lim_{N \to \infty} 2^{4/3} N^{4/3} p_{\lambda/\mu} \left(\frac{\gamma_1}{\gamma_1 + \gamma_2} t, \frac{\gamma_2}{\gamma_1 + \gamma_2} t, m, n \right) = \mathrm{p}_{\mathrm{f/s}} \left(\mathrm{t}, \mathrm{x}, \mathrm{y}; \frac{\gamma_1}{\gamma_1 + \gamma_2} \right)$$

Limit theorems

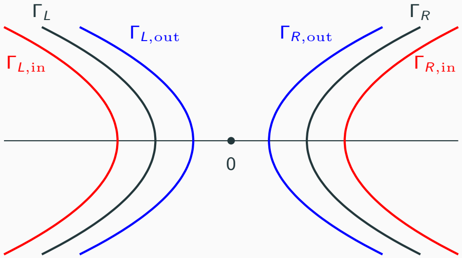


The function $\mathrm{p}_{\mathrm{f/s}}$

$$\mathrm{p}_{\mathrm{f/s}}(\mathrm{t},\mathrm{x},\mathrm{y};\gamma) := \oint_0 \frac{\mathrm{dz}}{2\pi\mathrm{i}(1-\mathrm{z})^2} \sum_{k_1,k_2\geq 1} \frac{1}{(k_1!k_2!)^2} \mathrm{T}_{k_1,k_2}(\mathrm{z};\mathrm{t},\mathrm{x},\mathrm{y};\gamma)$$

$$\begin{aligned}
&\mathrm{T}_{k_1,k_2}(\mathrm{z};\mathrm{t},\mathrm{x},\mathrm{y};\gamma)\\
&:= \prod_{i_1=1}^{k_1}\left(\frac{1}{1-\mathrm{z}}\int_{\Gamma_{\mathrm{L,in}}} \frac{\mathrm{d}\xi_{i_1}^{(1)}}{2\pi\mathrm{i}} - \frac{\mathrm{z}}{1-\mathrm{z}}\int_{\Gamma_{\mathrm{L,out}}} \frac{\mathrm{d}\xi_{i_1}^{(1)}}{2\pi\mathrm{i}}\right) \prod_{i_2=1}^{k_2}\int_{\Gamma_{\mathrm{L}}} \frac{\mathrm{d}\xi_{i_2}^{(2)}}{2\pi\mathrm{i}}\\
&\quad \prod_{i_1=1}^{k_1}\left(\frac{1}{1-\mathrm{z}}\int_{\Gamma_{\mathrm{R,in}}} \frac{\mathrm{d}\eta_{i_1}^{(1)}}{2\pi\mathrm{i}} - \frac{\mathrm{z}}{1-\mathrm{z}}\int_{\Gamma_{\mathrm{R,out}}} \frac{\mathrm{d}\eta_{i_1}^{(1)}}{2\pi\mathrm{i}}\right) \prod_{i_2=1}^{k_2}\int_{\Gamma_{\mathrm{R}}} \frac{\mathrm{d}\eta_{i_2}^{(2)}}{2\pi\mathrm{i}}\\
&\quad (1-\mathrm{z})^{k_2}\left(1-\frac{1}{\mathrm{z}}\right)^{k_1} \cdot \frac{\mathrm{f}_1(\boldsymbol{\xi}^{(1)};\gamma)\mathrm{f}_2(\boldsymbol{\xi}^{(2)};1-\gamma)}{\mathrm{f}_1(\boldsymbol{\eta}^{(1)};\gamma)\mathrm{f}_2(\boldsymbol{\eta}^{(2)};1-\gamma} \cdot \mathrm{H}(\boldsymbol{\xi}^{(1)},\boldsymbol{\eta}^{(1)};\boldsymbol{\xi}^{(2)},\boldsymbol{\eta}^{(2)})\\
&\quad \cdot \det\left[\delta(\xi_{j_2}^{(2)},-\eta_{i_2}^{(2)})\right] \cdot \prod_{\ell=1}^{2} \frac{\left(\Delta(\boldsymbol{\xi}^{(\ell)})\right)^2\left(\Delta(\boldsymbol{\eta}^{(\ell)})\right)^2}{\left(\Delta(\boldsymbol{\xi}^{(\ell)};\boldsymbol{\eta}^{(\ell)})\right)^2} \cdot \frac{\Delta(\boldsymbol{\xi}^{(1)};\boldsymbol{\eta}^{(2)})\Delta(\boldsymbol{\eta}^{(1)};\boldsymbol{\xi}^{(2)})}{\Delta(\boldsymbol{\xi}^{(1)};\boldsymbol{\xi}^{(2)})\Delta(\boldsymbol{\eta}^{(1)};\boldsymbol{\eta}^{(2)})}
\end{aligned}$$

The function $\mathrm{p}_{\mathrm{f/s}}$



The contours

The function $\mathrm{p}_{\mathrm{f/s}}$

The vectors $\boldsymbol{\xi}^{(\ell)} = (\xi_1^{(\ell)}, \cdots, \xi_{i_\ell}^{(\ell)})$ and $\boldsymbol{\eta}^{(\ell)} = (\eta_1^{(\ell)}, \cdots, \eta_{i_\ell}^{(\ell)})$ for $\ell \in \{1, 2\}$, the functions $\mathrm{f}_1, \mathrm{f}_2$ are defined by

$$\mathrm{f}_1(\zeta;\gamma) := \exp\left(-\frac{\gamma}{6}\zeta^3 - \frac{1}{4}\mathrm{x}\zeta^2 + \frac{1}{2}\left(\mathrm{y} + \gamma\mathrm{t}\right)\zeta\right),$$

$$\mathrm{f}_2(\zeta;\gamma) := \exp\left(-\frac{\gamma}{6}\zeta^3 + \frac{1}{4}\mathrm{x}\zeta^2 + \frac{1}{2}\left(-\mathrm{y} + \gamma\mathrm{t}\right)\zeta\right),$$

and the function H is defined by

$$\mathrm{H}(\boldsymbol{\xi}^{(1)}, \boldsymbol{\eta}^{(1)}; \boldsymbol{\xi}^{(2)}, \boldsymbol{\eta}^{(2)}) = \frac{1}{12}\mathrm{S}_1^4 + \frac{1}{4}\mathrm{S}_2^2 - \frac{1}{3}\mathrm{S}_1\mathrm{S}_3$$

with

$$\begin{aligned} \mathrm{S}_\ell &= \mathrm{S}_\ell(\boldsymbol{\xi}^{(1)}, \boldsymbol{\eta}^{(1)}; \boldsymbol{\xi}^{(2)}, \boldsymbol{\eta}^{(2)}) \\ &= \sum_{i_1=1}^{k_1}\left(\left(\xi_{i_1}^{(1)}\right)^\ell - \left(\eta_{i_1}^{(1)}\right)^\ell\right) - \sum_{i_2=1}^{k_2}\left(\left(\xi_{i_2}^{(2)}\right)^\ell - \left(\eta_{i_2}^{(2)}\right)^\ell\right). \end{aligned}$$

The function $\mathrm{p}_{\mathrm{f/s}}$

Corollary (?)
Let $\mathcal{A}_2$ and $\mathcal{A}_1$ be independent Airy$_2$ and Airy$_1$ processes respectively. Denote $\hat{\mathcal{A}}_2(x) = \mathcal{A}_2(x) - x^2$. Denote

$$\mathcal{T} = \mathrm{argmax}_{\mathrm{x}} \left((\gamma/2)^{1/3} \hat{\mathcal{A}}_2 \left(\frac{\mathrm{x}}{2^{4/3}\gamma^{2/3}} \right) + 2(1-\gamma)^{1/3} \mathcal{A}_1 \left(\frac{\mathrm{x}}{4(1-\gamma)^{2/3}} \right) \right).$$

Then $\mathrm{p}_{\mathrm{f/s}}\left(\mathrm{h}_1 + \mathrm{h}_2, \mathrm{x}, (1-\gamma)\mathrm{h}_1 - \gamma \mathrm{h}_2; \gamma\right)$ is the joint probability density function at

$$\left((\gamma/2)^{1/3} \hat{\mathcal{A}}_2 \left(\frac{\mathrm{x}}{2^{4/3}\gamma^{2/3}} \right), 2(1-\gamma)^{1/3} \mathcal{A}_1 \left(\frac{\mathrm{x}}{4(1-\gamma)^{2/3}} \right), \mathcal{T} \right) = (\mathrm{h}_1, \mathrm{h}_2, \mathrm{x}).$$

5. Applications

Rigidity of the geodesic

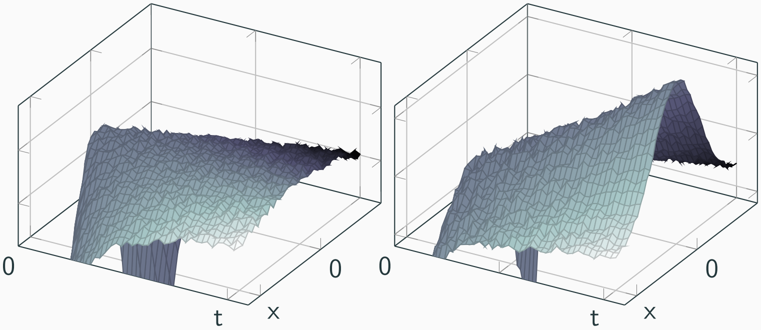
Denote $\mathcal{L}(y,s;x,t)$ the directed landscape, the limiting four-parameter random field of the directed last passage percolation. We fix two points $(0,0)$ and $(0,1)$. Denote $\Pi(s)$ the geodesic from $(0,0)$ to $(0,1)$. We also denote $\mathcal{L}(s) = \mathcal{L}(0,0;\Pi(s),s)$

Theorem (Liu21)
The random variables

$$\frac{2\Pi(s)\mathcal{L}(1)^{1/4}}{\sqrt{s(1-s)}}, \frac{\mathcal{L}(s) - s\mathcal{L}(1)}{\sqrt{s(1-s)}\mathcal{L}(1)^{1/4}}$$

conditioned on $\mathcal{L}(1) \to \infty$*, converge to two independent standard Gaussian random variables in distribution.*

Rigidity of the geodesic



Two **imaginary** figures of the directed landscape $\mathcal{L}(0, 0; x, t)$

Other related works in progress

Conditioned on $\mathcal{L}(0,0;0,1) = L$ is large, what is the limiting behavior of $\mathcal{L}(0,0;x,t)$?

(a) When $t < 1$, there is a limiting field $L^{-1/4}(\mathcal{L}(0,0;xL^{-1/4},t) - tL)$ whose finite dimensional distribution along the time direction can be described in terms of two independent Brownian bridges. [Liu-Wang21+]

(b) When $t > 1$, $\mathcal{L}(0,0;0,t) - \mathcal{L}(0,0;0,1)$ behaves like a Tracy-Widom random variable, plus explicit perturbations. [Nissim-Zhang21+]

6. Open problems

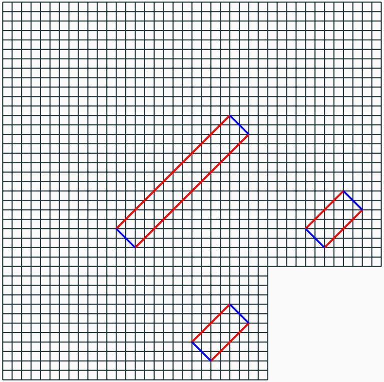
1. Fluctuations of the geodesic $\Pi(s)$ when $\mathcal{L}(1) \to -\infty$

When $\mathcal{L}(1) \to \infty$, we know that $\Pi(s)$ becomes very rigid and the limiting fluctuations are given by the Gaussian distributions.

What happens when $\mathcal{L}(1) \to -\infty$? Intuitively, the geodesic $\Pi(s)$ will have a higher order of fluctuation. It is related to the left tail of our formula, which seems much more difficult.

2. Coalescence of geodesics

In the random melting skew Young diagram model, suppose $\mu = (0)$ and $\lambda = (\underbrace{N, \cdots, N}_{N - N^{2/3}}, \underbrace{N - cN^{2/3}, \cdots, N - cN^{2/3}}_{N^{2/3}})$.



Intuitively this problem is equivalent to the coalanscence of two geodesics from the northwest corner to the southeast corners in the corresponding DLPP model.

The key is to analyze the kernel $\hat{K}_{\lambda}^{(N)}(v, u)$, but it seems not easy!

Thank you!