On Finite-Rank Non-Hermitian Deformations of Random Matrix Ensembles

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Dedicated to the memory of **Konstantin Efetov** (1950-2021)

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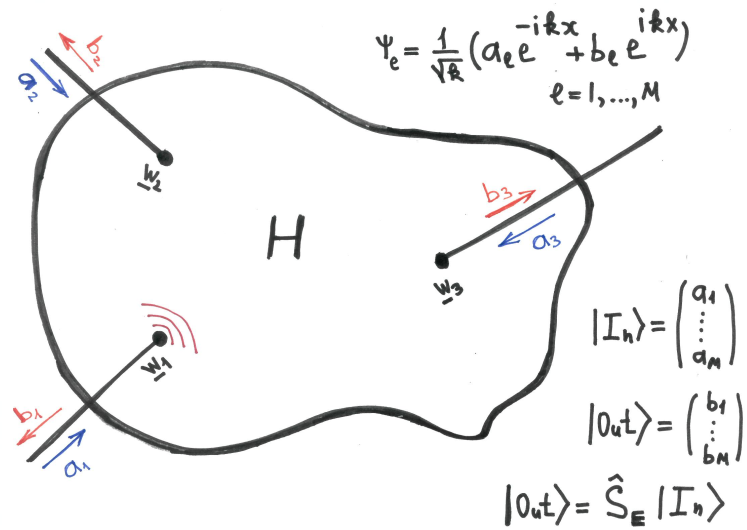
RMT approach to chaotic wave scattering:

H $=$ **- Laplacian (+ Dirichlet B.C.)**



Heidelberg Program [**Verbaarschot, Weidenmüller, Zirnbauer '85**]:

To describe **generic/universal** properties of the associated **scattering** system by replacing H for the "inner" region supporting classically chaotic dynamics with **random GOE/GUE matrix** H of size $N \gg 1$, and the couplings to M scattering channels with M vectors $\mathbf{w}_a$, $a = 1, \ldots, M$.

In this framework, **Sokolov-Zelevinsky** '88 demonstrated that the **generic/universal** features of statistics of the **poles** of the scattering matrix $S(E)$ in the complex energy plane (a.k.a. **resonances**) can be modelled by N complex eigenvalues z_n in the lower half-plane $\Im z_n \leq 0, \forall n = 1, \ldots, N$ of the effective **non-selfadjoint** random matrix "Hamiltonian":

$$\mathcal{H} = H - i\Gamma, \quad \Gamma := \sum_a \mathbf{w}_a \otimes \mathbf{w}_a^* = WW^* \geq 0 - \text{ rank M}$$

Note 1: The effective **non-Hermitian** Hamiltonian

$$\mathcal{H} = H - i\,\Gamma,\ \Gamma = WW^* \geq 0 - \ \textbf{rank M}$$

is **non-normal**, hence complex eigenvalues $z_i = X_i - iY_i$ come with two sets of eigenvectors: **"right"** $\boldsymbol{r}_i$ and **"left"** ones $\boldsymbol{l}_i$ satisfying

$\mathcal{H}\boldsymbol{r}_i = z_i\boldsymbol{r}_i$ and $\mathcal{H}^*\boldsymbol{l}_i = \overline{z}_i\boldsymbol{l}_i$ as well as the **bi-orthogonality** relation $(\boldsymbol{l}_i^*\boldsymbol{r}_j) = \delta_{ij}$.

The corresponding **non-orthogonality overlap** matrix

$$\mathcal{O}_{mn} = (\boldsymbol{l}_m^*\boldsymbol{l}_n)(\boldsymbol{r}_n^*\boldsymbol{r}_m)$$

shows up in various experimental observables of wave-chaotic systems, such as e.g. **decay laws Savin, Sokolov** '97, **excess noise** in laser resonators (**Schomerus et al.** '00), in **sensitivity** of scattering to small perturbations (**YF, Savin** '12), in **transmission** and **reflection** statistics, **Davy, Genack** '19,**YVF, Osman** '21

Characterizing statistics of $\mathcal{O}_{mn}$ presents a serious challenge.

Note 2: Finite-rank deformations of random **Hermitian** matrices are closely related to finite-rank deformations of random **unitary** matrices, such as **truncations**.

Note 3: The matrices $H - i\,\Gamma$ at fixed M are **weakly non-Hermitian** when $N \to \infty$: typically $\Im z_i = Y_i \sim \Delta = O(1/N)$, where $\Delta = (N\rho_{sc}(x))^{-1}$ is the eigenvalue spacing. Distribution of $\Im z_i$ (aka **resonance widths**) is an interesting problem.

Example of rank-one non-Hermitian deformation:

Consider

$\mathcal{H} = H_N - i\gamma \mathbf{e}\otimes\mathbf{e}^T$ with $N\times N$ matrix $H_N \in GOE/GUE/\beta - Hermite$

The **Joint Probability Density** (JPD) of complex eigenvalues z_i for $\mathcal{H}$

in the half-plane $\Im z_j \le 0, \forall j = 1,\dots,N$ is known from **N. Ullah** '69, Sokolov-Zelevinsky '88, Seba-Stöckmann 98, **Kozhan** '17.

$$\mathcal{P}_z\{z_i\} = \frac{1}{h_{\beta,N}}\, e^{-\frac{\beta N}{4}\sum_{i=1}^N (Re\, z_i)^2} \prod_{1\le j<k\le N} |z_j - z_k|^2$$
$$\times \prod_{j,k=1}^N |z_j - \overline{z}_k|^{\frac{\beta}{2}-1}\, \delta\left(\sum_{i=1}^N Im\, z_j + \gamma\right)$$

where $h_{\beta,N} = 2^{N\left(\frac{\beta}{2}-1\right)}\gamma^{\frac{\beta N}{2}} e^{\frac{\beta N}{4}\gamma^2} Z_{\beta,N}\, C_{\beta,N}$.

In the case $\beta = 2$ the eigenvalues asymptotically, for $N \gg 1$, form a **determinantal process** in the lower half of the complex plane, and all their finite-order correlation functions (aka marginal densities) can be found via a certain kernel [**YVF, Khoruzhenko** '99]. Moreover, these results can be extended for any **fixed**-rank deformations $1 \le M < \infty$ and $N \to \infty$.

All $\beta \neq 2$, including the cases $\beta = 1\&4$, present a serious challenge.

$\beta = 2$, fixed rank deformation $-i\Gamma := -i\sum_{c=1}^{M} \gamma_c \mathbf{e}_a \otimes \mathbf{e}_a^*$:

Define " **renormalized** coupling strengths" $g_c = \frac{1}{2}\left(\gamma_c + \frac{1}{\gamma_c}\right)$ for all $c = 1, \ldots, M$. Then

$$\lim_{N\to\infty} \frac{1}{N^{2n}} R_n(z_1 = x + \frac{\zeta_1}{N}, \ldots, z_n = x + \frac{\zeta_n}{N}) \to \det\{K(\zeta_j, \zeta_k)\}$$
$$K(\zeta_1, \zeta_2) = F^{1/2}(\zeta_1) F^{1/2}(\zeta_2) \int_{-1}^{1} e^{-i\pi\rho\lambda(\zeta_1 - \bar{\zeta}_2)} \prod_{c=1}^{M} (g_c + \pi\rho\lambda)\, d\lambda$$

where $\rho := \frac{1}{4\pi}\sqrt{4 - x^2}$ is the mean density of GUE eigenvalues around E and

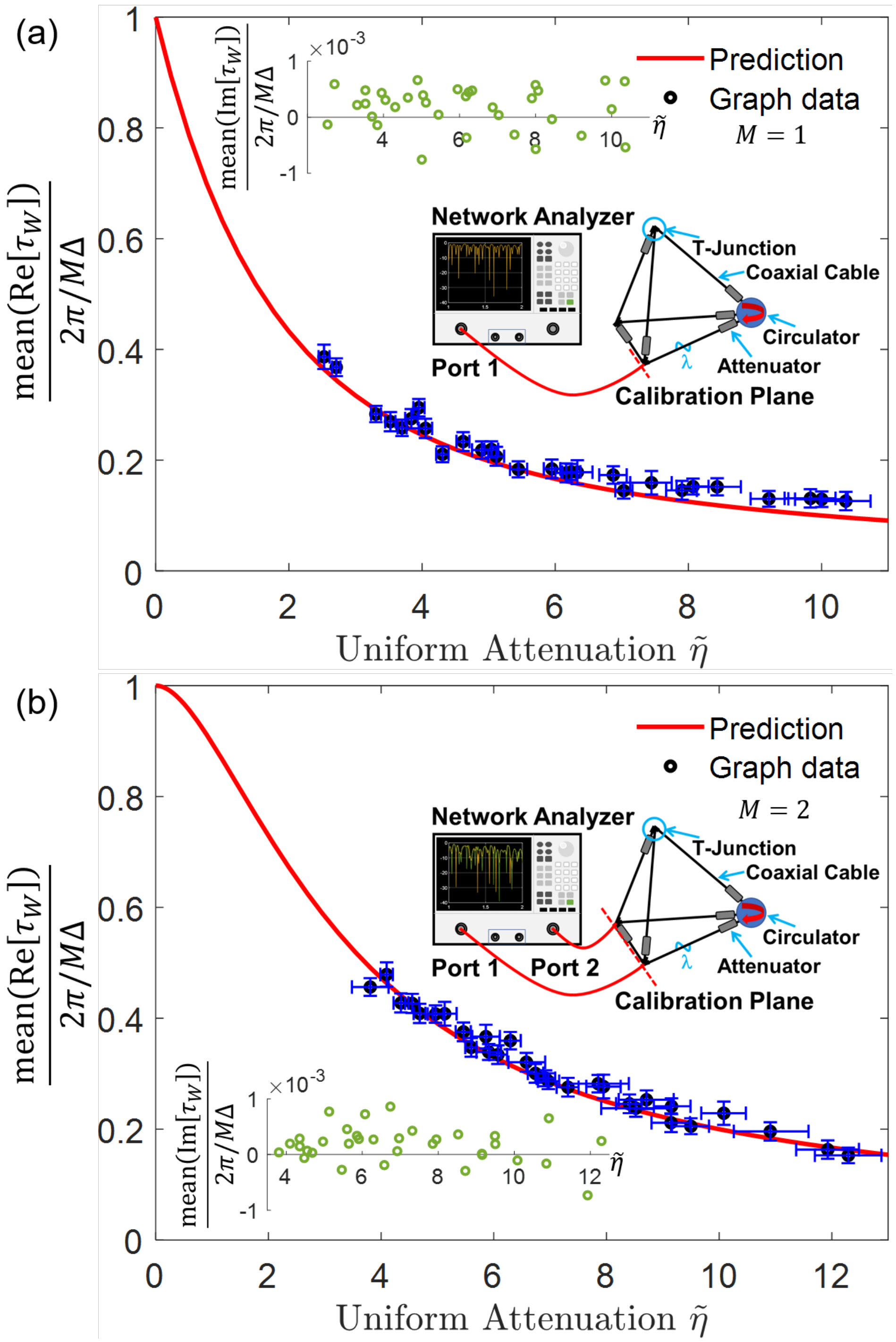
$$F(\zeta) = \sum_{c=1}^{M} \frac{e^{-2g_c|\mathrm{Im}\zeta|}}{\prod_{s\neq c}^{M}(g_c - g_s)}$$

In particular, the probability density of the scaled **imaginary parts** $y_n = \pi \Im z_n / \Delta$ is given for M equivalent channels with $g_1 = \ldots = g_M \equiv g$ by (**YF, Sommers**'96)

$$\mathcal{P}_M^{(\beta)}(y) = \frac{(-1)^M}{(M-1)!} y^{M-1} \frac{d^M}{dy^M}\left\{ e^{-yg}\left(\frac{\sinh y}{y}\right)\right\}$$

For $\gamma \neq 1$ we have the exponential tail: $\mathcal{P}_M^{(\beta)}(y \gg 1) \propto e^{-(g-1)y}$.
In contrast, for the **perfect coupling** case $g = 1$ the **power-law tail** emerges:

$$\mathcal{P}_M^{(\beta)}(y >> 1) \propto 1/y^2.$$



Experimental data

L Chen, S. M. Anlage & YVF

arXiv:2106.15469

vs. theoretical prediction

YVF, H.-J. Sommers ’96

$\beta = 1$, fixed $M < \infty$:

For $\beta = 1$ the eigv. density in the complex plane can be found by **Efetov SUSY** approach (**Sommers, YF, Titov**'99). Defining $y_n = \pi \Im z_n / \Delta$ one finds for $M = 1$

$$\mathcal{P}^{(\beta=1)}_{M=1}(y) = \frac{1}{4\pi} \frac{d^2}{dy^2} \int_{-1}^{1} (1-\lambda^2) e^{2\lambda y} (g-\lambda) \mathcal{F}(\lambda, y)\, d\lambda$$

where

$$\mathcal{F}(\lambda, y) = \int_g^\infty dp_1 \frac{e^{-yp_1}}{(\lambda-p_1)^2 \sqrt{(p_1^2-1)(p_1-g)}} \int_1^g dp_2 \frac{e^{-yp_2}}{(\lambda-p_2)^2 \sqrt{(p_2^2-1)(g-p_2)}}$$

and even more complicated expressions for $M > 1$.

Very recently an alternative method for $M = 1$ was proposed in **YVF, Osman**'21 and is based on exploiting the known **eigenvalue JPD** with **non-Efetov SuSy**

$$\mathcal{P}^{\beta=1}_N(z_1, \ldots, z_N) \propto \frac{e^{-\frac{N}{4}\left(\gamma^2 + \sum_{j=1}^N (\Re z_j^2)\right)}}{\gamma^{\frac{N}{2}-1}} \prod_{j,k=1}^N \frac{1}{\sqrt{|z_j - z_k^*|}} \prod_{j<k}^N |z_j - z_k|^2 \delta\Big(\sum_{j=1}^N \Im z_j + \gamma\Big)$$

which gives instead

$$\mathcal{P}^{(\beta=1)}_{M=1}(y) = \frac{1}{4\sqrt{2}} e^{-gy}\, \mathbb{L}_1 \int_1^\infty da\, e^{-gay} \frac{(a-1)}{\sqrt{a+1}} I_0\left(y\sqrt{(g^2-1)(a^2-1)}\right)$$

where $\mathbb{L}_1$ is the following differential operator:

$$\mathbb{L}_1 = 2\sinh 2y - \left(\cosh(2y) - \frac{\sinh 2y}{2y}\right)\left(\frac{3}{y} + 2\frac{d}{dy}\right)$$

Numerically the two expressions are **indistinguishable**, but we haven't found a way to prove it yet.

Diagonal non-orthogonality factors for rank-one non-Hermitian deformations:

Theorem **YVF, M. Osman** , '21

Consider $\mathcal{H} = H - i\gamma \mathbf{e} \otimes \mathbf{e}^T$*, with* $H \in GUE$ *or* $H \in GOE$ *and define the non-orthogonality factor* $\mathcal{O}_n = (\boldsymbol{l}_n^* \boldsymbol{l}_n)(\boldsymbol{r}_n^* \boldsymbol{r}_n)$ *for eigenvalues* z_n*. Define the probability density of* $t = O_{nn} - 1$ *corresponding to eigenvalues in the vicinity of a point* $z = X - iY,\ Y > 0$ *in the complex plane:*

$$\mathcal{P}(t; z) = \left\langle \frac{1}{N} \sum_{i=1}^{N} \delta(O_{nn} - 1 - t)\delta(z - z_n) \right\rangle$$

Then for $H \in GUE$ *as* $N \to \infty$ *the limiting density*
$\mathcal{P}_y^{(2)}(t) := \lim_{N\to\infty} \frac{1}{\pi\rho N} \mathcal{P}(t; z = X - i\frac{y}{\pi\rho N})$ *takes the following form*

$$\mathcal{P}_y^{(2)}(t) = \frac{16}{t^3}\, e^{-2gy}\, \mathbb{L}_2\, e^{-2gy\left(1+\frac{2}{t}\right)} I_0\left(\frac{4y}{t}\sqrt{(g^2-1)(1+t)}\right)$$

where we defined $g = \frac{1}{2\pi\rho_{sc}(x)}\left(\gamma + \frac{1}{\gamma}\right)$*,* $I_0(x)$ *stands for the modified Bessel function and* $\mathbb{L}_2$ *is a differential operator acting on functions* $f(y)$ *as*

$$\mathbb{L}_2\, f(y) = \left\{1 + \left(\frac{\sinh 2y}{2y}\right)^2 + \frac{1}{2y}\left(1 - \frac{\sinh 4y}{4y}\right)\frac{d}{dy} + \frac{1}{4}\left(\left(\frac{\sinh 2y}{2y}\right)^2 - 1\right)\frac{d^2}{dy^2}\right\} y^2 f(y).$$

For $H \in GOE$ we have a similar result:

$$\mathcal{P}_y^{(1)}(t) = \frac{1}{2}\frac{e^{-gy}}{\sqrt{t^5(1+t)}}\, \mathbb{L}_1\, e^{-gy\left(1+\frac{2}{t}\right)} I_0\left(\frac{2y}{t}\sqrt{(g^2-1)(1+t)}\right)$$

where

$$\mathbb{L}_1 = 2\sinh 2y - \left(\cosh 2y - \frac{\sinh 2y}{2y}\right)\left(\frac{3}{y} + 2\frac{d}{dy}\right)$$

Main object to be evaluated:

$$\mathcal{M}_p^{(\beta)}(z) = \left\langle \frac{\left[\det(z-\tilde{\mathcal{H}})\det(\overline{z}-\tilde{\mathcal{H}}^\dagger)\right]^{p+1}}{\left[\det(z-\tilde{\mathcal{H}}^\dagger)\det(\overline{z}-\tilde{\mathcal{H}})\right]^{p+1-\frac{\beta}{2}}} \right\rangle$$

For $\beta = 2$ can be reduced to **YF, Strahov**'03 or **Borodin, Strahov**'04 formulas. For $\beta = 1$ can be reduced to a result of **YF, Nock**'15.

Off-diagonal non-orthogonality correlator, $\beta = 2$:

One can also study the off-diagonal non-orthogonality factors $\mathcal{O}_{mn} = (\boldsymbol{l}_m^* \boldsymbol{l}_n)(\boldsymbol{r}_n^* \boldsymbol{r}_m)$. Consider **microscopic** eigenvalue separation $\Re(z_a - z_b)/2 = \Omega \sim \Delta = O(1/N)$. Introducing $\omega = \frac{\pi \Re(z_a - z_b)}{\Delta}$ and $y_{a,b} = \frac{\pi \Im z_a}{\Delta}$ one gets (**YVF, Mehlig**'02):

$$O(z_a, z_b) := \left\langle \frac{1}{N} \sum_{n \neq m} \mathcal{O}_{nm} \; \delta(z_a - z_n) \; \delta(z_b - z_m) \right\rangle_{H \in GUE}$$

$$= N(\pi \rho_{sc}(x)/\Delta)^2 e^{-2g(y_a+y_b)} \det \begin{pmatrix} F(i\omega + y_a - y_b) & F(i\omega + y_a + y_b) \\ F(i\omega - y_a - y_b) & F(i\omega - y_a + y_b) \end{pmatrix}$$

with

$$F(u) = 2\left(g + \frac{d}{du}\right) \frac{\sinh u}{u}.$$

Most probably indication of a **determinantal structure** for conditional overlaps holds, similar to one found for complex Ginibre in **Akemann, Tribe, Tsareas, and Zaboronski**'19.

Scattering in quasi $1D$ systems with Anderson localization:

Physical system: a **disordered wire** of length L with random potential inside, characterized by a **localization length** ξ, and attached to the scattering channels at one of edges.

Mathematical model: a random **banded matrix** of size $N \sim L$ and the **localization length** $\xi \sim b^2 \gg 1$ deformed by adding the anti-Hermitian diagonal matrix $-i\Gamma := -i\sum_{c=1}^{M} \gamma_c \mathbf{e}_a \otimes \mathbf{e}_a^*$.

For the "complete localization" limit $N/b^2 \to \infty$ the density of imaginary parts $\Im z_n$ of complex eigenvalues can be calculated in the framework of the **Efetov SUSY** approach.
Defining Δ_ξ to be the eigv. spacing at the sample of localization length ξ, the probability density of the properly re-scaled $y_n = \pi \Im z_n / \Delta_\xi$ for M **perfectly coupled** channels $\gamma_c = 1, c = 1, \ldots, M$ is given by (**YVF, M. Skvortsov, K. Tikhonov** '21):

$$\rho(y) = -\frac{4}{\pi^2\kappa}\frac{\partial}{\partial\kappa}\left[\frac{1}{\kappa}\sum_{n=0}^{M-1} K_n(\kappa) I_{n+1}(\kappa)\right] \text{ where } \kappa = \sqrt{\frac{8y}{\pi}}$$

Analysis at $M \gg 1$ shows that

$$\rho(\Im z) \sim \begin{cases} \Delta_\xi/\Im z & \text{for } \Im z \ll \Delta_\xi \\ (\Delta_\xi/\Im z)^{3/2} & \text{for } \Delta_\xi \ll \Im z \ll M^2\Delta_\xi \\ M(\Delta_\xi/\Im z)^2 & \text{for } \Im z \gg M^2\Delta_\xi \end{cases}$$

The results agree with numerics for **banded matrices** and with "**quantum kicked rotator**" behaviour observed in **Borgonovi, Guarneri, and Shepelyansky** '91

Challenge: to describe $\rho(\Gamma)$ close to the **Anderson localization** transition.