Introduction to Water Waves
Lecture 3

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Energy estimates for gravity waves

Control parameters:

s = scaling parameter
s = 0 → scaling

$$A_s = \|(W_\alpha, |D|^{\frac{1}{2}} Q_\alpha)\|_{BMO^s}$$

1. Classical bounds [ABZ'11]

$$\frac{d}{dt}E(t) \lesssim A_{\frac{1}{2}+\epsilon} E(t)$$

→ $\|\nabla v\|_{L^\infty}$

2. Cubic energy bounds [HIT'14, modified energy]

$$\frac{d}{dt}E(t) \lesssim_{A_0} A_0 A_{\frac{1}{2}} E(t)$$

→ cubic
→ scale invariant
→ $\nabla v \in BMO$

3. Balanced cubic energy bounds [AIT'19, (better) modified energy]

$$\frac{d}{dt}E(t) \lesssim_{A_0} A^2_{\frac{1}{4}} E(t)$$

$D^{3/4} v \in BMO$

- reduction to paralinearization
- refined, variable coefficient normal form analysis for balanced frequency interactions
- modified energy for paralinearization

Four water wave equations

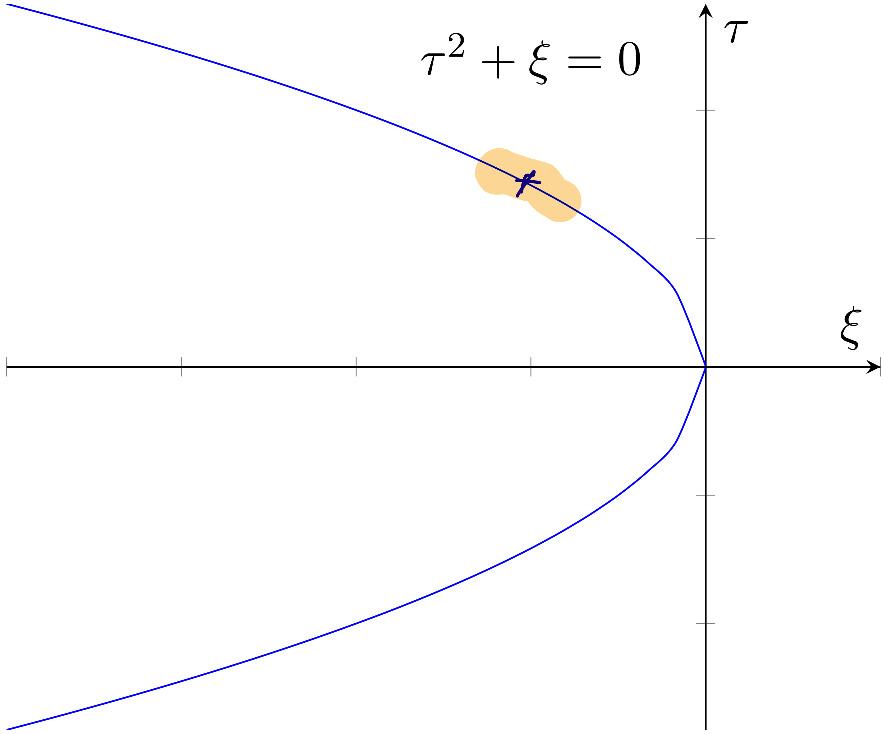
1. **Gravity waves in deep water** (g)
   - infinite bottom, gravity, no surface tension (long waves)
   - (1-D cubic) NLS approximation for frequency localized data

2. **Capillary waves in deep water** (t)
   - infinite bottom, surface tension, no gravity (short waves)
   - NLS approximation for frequency localized data

3. **Constant vorticity gravity waves in deep water** (v)
   - infinite bottom, no surface tension, gravity, constant vorticity (tides)
   - Benjamin-Ono approximation at low frequency

4. **Gravity waves in shallow water** (b)
   - finite bottom, no surface tension, gravity
   - KdV approximation at low frequency

Collaborators: **Mihaela Ifrim** (U. Wisconsin), John Hunter (UC Davis), Benjamin Harrop-Griffiths (UCLA), Thomas Alazard (ENS Saclay), Herbert Koch (Bonn), Albert Ai (U. Wisconsin), WW-group of graduate students in Berkeley&Madison



Dispersion relation for gravity waves in deep water (g)

Cubic NLS approximation:

$$(i\partial_t + \partial_x^2)u = \pm|u|^2u$$

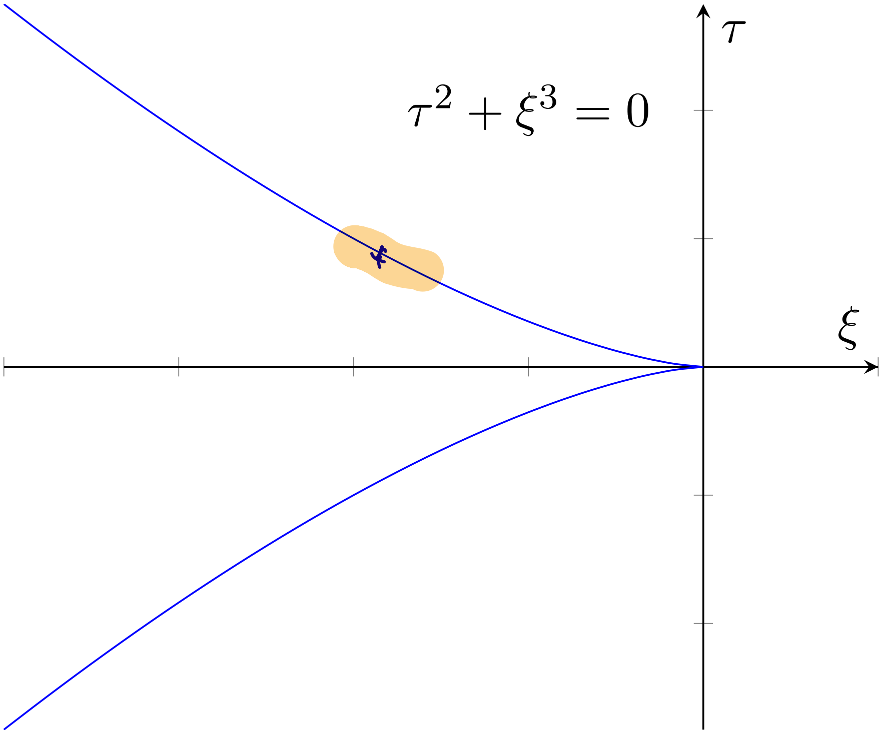


Figure: Dispersion relation for capillary waves in deep water (t)

Cubic NLS approximation:

$$(i\partial_t + \partial_x^2)u = \pm|u|^2 u$$

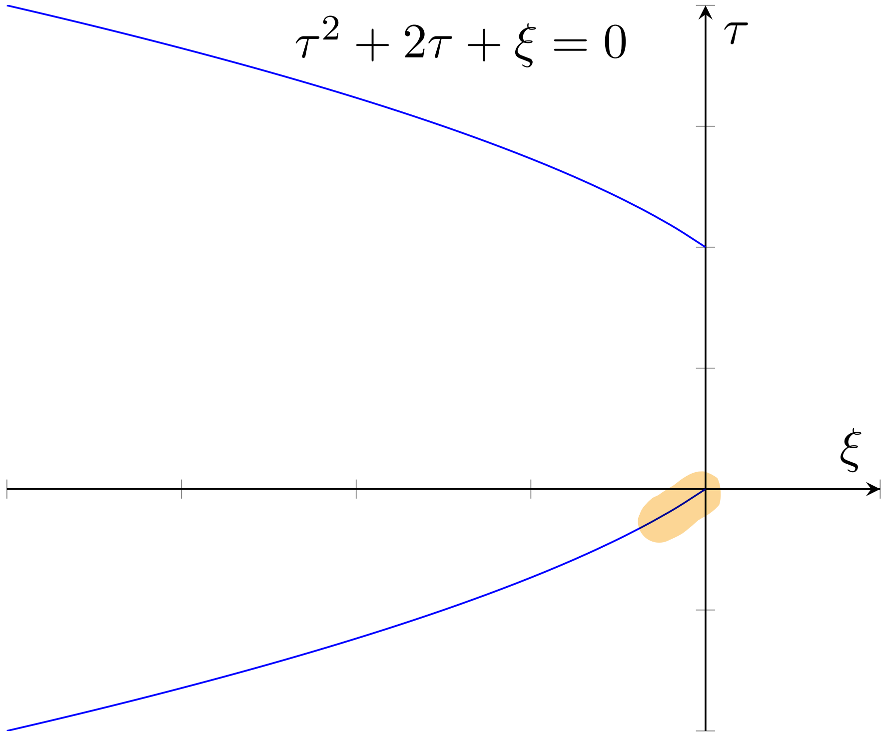


Figure: Dispersion relation for constant vorticity (v)

Benjamin-Ono approximation:

$$(\partial_t + H\partial_x^2)u = uu_x$$

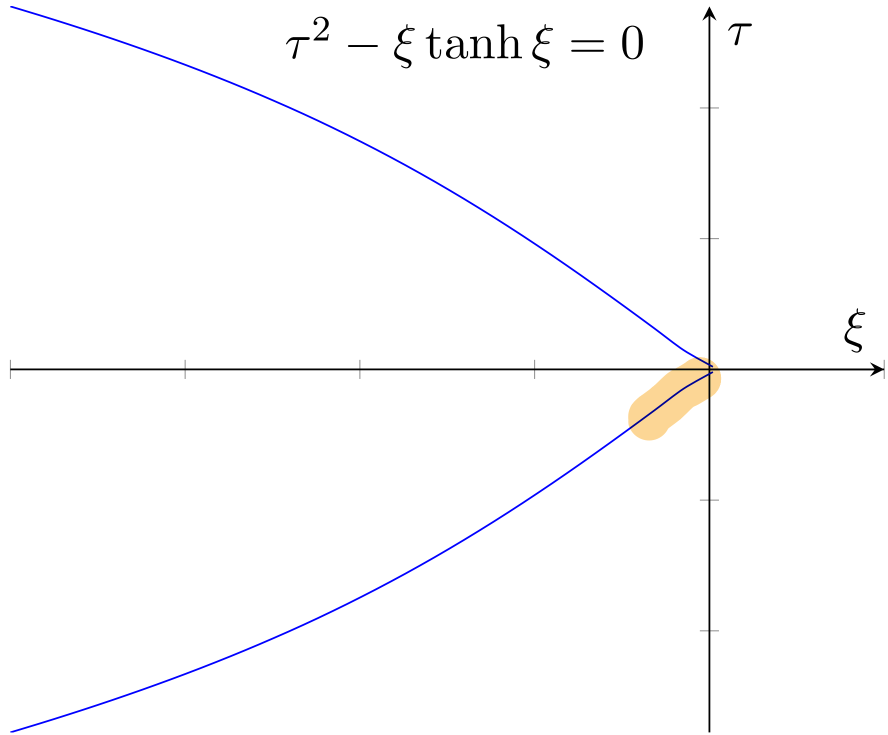


Figure: Dispersion relation, gravity waves in shallow water (b)

KdV approximation:

$$(\partial_t + \partial_x^3)u = 6uu_x$$

Cubic lifespan bounds

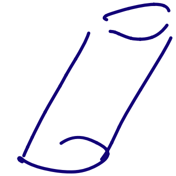
Theorem

Consider the two dimensional differentiated water wave equation with initial data of size ϵ. Then the solutions have a lifespan of at least

$$T_\epsilon \approx \epsilon^{-2}$$

- The result applies to all four models (g), (v) (t) and (b).
- The result applies equally in periodic and non-periodic setting.
- Proof idea: *quasilinear modified energy method*
- Bounds for all higher norms propagate on same timescale.
- Additional difficulty for water waves, due to the fact that the system is degenerate hyperbolic. Because of this, the modified energy needs to be in the diagonal variables.
- Related work of Wu (g), Ionescu-Pusateri (g), (t), Berti-Delort (g)/(t).

Cubic NLS approximation



Theorem (Ifrim-T '18)

Consider the two dimensional differentiated water wave equation (g) with wave packet initial data of size $\epsilon^{\frac{1}{2}}$. *Then the solutions have a lifespan of at least*

$$T_\epsilon \approx M\epsilon^{-2}$$

and are well-approximated by a cubic NLS flow.

- Wave packet data = localized near a frequency ξ_0 on scale $\delta\xi = \epsilon$.
- M= NLS time, should be large $M \approx \log \epsilon$.
- Better corelation between water wave and NLS after normal form transformation.
- Related work of Wu, Schneider, Dull, etc

Global solutions for water waves on the line

Question: Given small and localized data, are the solutions global in time ? If so what is their asymptotic behavior ?

Two different patterns:

1. Dispersion wins: The solution exhibits linear like dispersive decay,

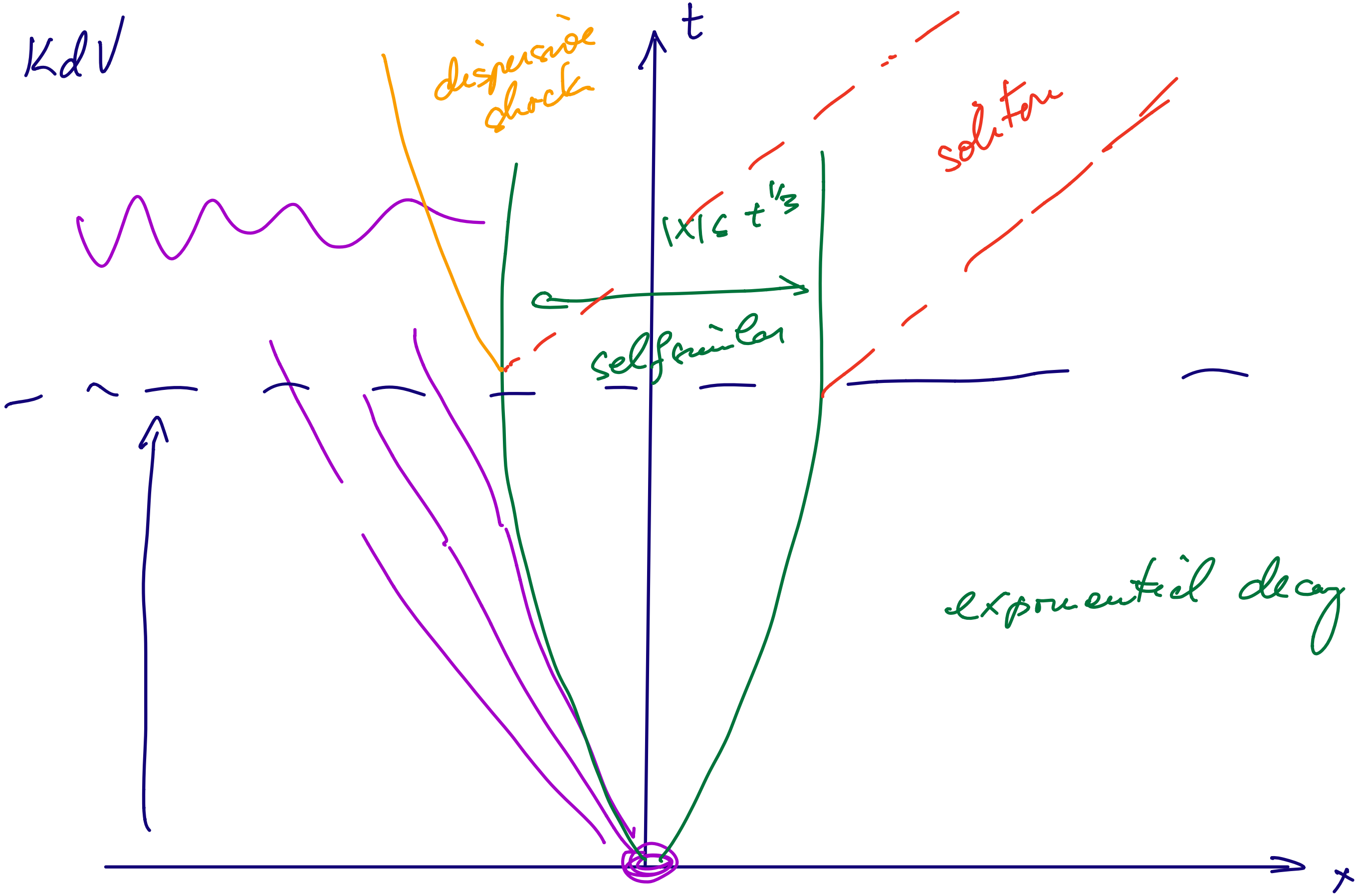
$$|u(x,t)| \lesssim \frac{1}{\sqrt{t}}$$

small

2. Nonlinearity balances the dispersion: Solitary waves form

Soliton resolution conjecture: Given small and localized data, all the solutions are global and resolve into a superposition of dispersive waves and one or more solitons.

Global solutions for water waves on the line



Linear vs. modified scattering

Linear equation:

$$iu_t = A(D_x)u, \qquad u(0) = u_0$$

Linear scattering (U is called scattering profile)

$$u(t,x) \approx U(v)\frac{1}{\sqrt{t}}\, e^{it\phi(v)}, \qquad v = \frac{x}{t}$$

Nonlinear equation:

$$iu_t = A(D_x)u + \lambda u|u|^2, \qquad u(0) = u_0$$

Trying ansatz

$$u(t,x) \approx \gamma(t,v)\frac{1}{\sqrt{t}}\, e^{it\phi(v)},$$

yields the asymptotic equation

$$i\partial_t \gamma \approx \frac{\lambda}{t}\gamma|\gamma|^2 \qquad \text{or equivalently} \qquad i\partial_s \gamma \approx \lambda\gamma|\gamma|^2, \qquad s = \log t$$

New asymptotic profile W

$$\gamma(s,v) \approx W(v)e^{i\lambda c(v) s|W|^2}$$

Key difficulty: make a good choice for asympt. profile γ.

Two objectives, to show that

1. γ is a good approximation for u
2. γ is an approximate solution to the asymptotic equation.

Earlier work:

- Define γ pointwise (Lindblad-Soffer, etc.)

$$u(t,x) = \gamma(t,v)\frac{1}{\sqrt{t}}\, e^{it\phi(v)}, \qquad v = x/t$$

- Define γ in Fourier space (Hayashi-Naumkin, etc)

$$\hat{u}(t,\xi) = \gamma(t,\xi)e^{ita(\xi)}, \qquad \xi = \xi_v$$

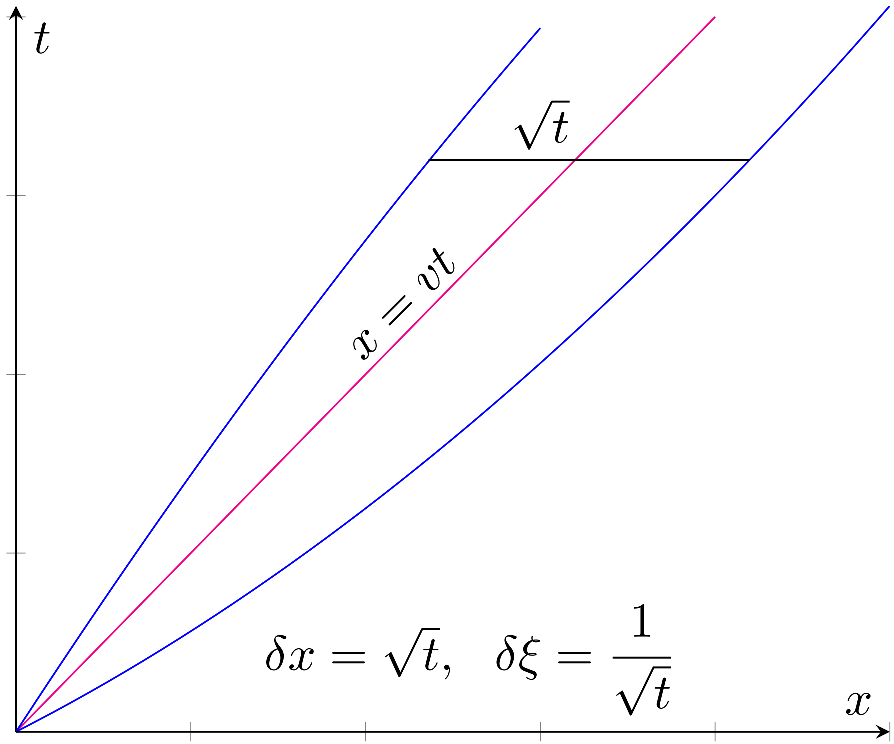
$v = v_\xi = a'(\xi)$

- Use the scattering transform (Deift-Zhou, integrable systems)

A better way: Testing by wave packets

[Ifrim-T. ’14], balances better the linear and nonlinear errors in asymptotic equation.

$$\gamma(t,v) = \langle u, \mathbf{u}_v \rangle, \qquad \mathbf{u}_v = \chi\left(\frac{x - vt}{\sqrt{t}}\right) e^{it\phi(v)}$$



Global water waves for small localized data

Theorem (Ifrim-T '14, Ai-Ifrim-T. '20)

Assume that the initial data for the water wave equation (g),(t) has size

$$\|(W,Q)(0)\|_{\mathcal{WH}} \lesssim \epsilon$$

Then the solution exists globally in time, with energy bounds

$$\|(W,Q)(t)\|_{\mathcal{WH}} \lesssim \epsilon t^{C\epsilon^2}$$

and pointwise decay

$$\|(W,Q)(t)\|_\infty \lesssim \frac{\epsilon}{\sqrt{t}}$$

- $\mathcal{WH}$ is a (time dependent) weighted localized L^2 Sobolev norm.
- Result includes modified scattering
- (g): Wu (ag), Ionescu-Pusateri, Alazard-Delort
  Simpler, shorter proofs by Hunter-Ifrim-T. (ag), Ifrim-T.
  Almost optimal result by Ai-Ifrim-T.
- (t): Ifrim-T., further work by Ionescu-Pusateri

Proof idea: bootstrap argument

Make the bootstrap assumption

$$\|(W,Q)(t)\|_\infty \lesssim \frac{C\epsilon}{\sqrt{t}}$$

Then proof in two steps:

- Cubic energy estimates (modified energy):

$$\frac{d}{dt}E_{\mathcal{WH}}(W,Q) \lesssim \|(W,Q)\|^2_{L^\infty} E_{\mathcal{WH}}(W,Q)$$

$\rightarrow \frac{C^2\varepsilon^2}{t} \rightarrow t^{C^2\varepsilon^2}$

  both for (W,Q) and for $S(W,Q)$. → linearized eqn.

- Pointwise estimates (improving the bootstrap assumption)

$$\|(W,Q)(t)\|_\infty \lesssim \frac{\epsilon}{\sqrt{t}}$$

  via the asymptotic profile γ, using the asymptotic equation.

No solitary waves in deep water

Theorem (Ifrim-T '18)

For the two dimensional water wave equation (g) and (t) there are no solitary waves.

- Prior partial results for (g) by Craig, Hur, Sun.
- For gravity waves the result also forbids crested waves (e.g. like the Stokes wave).
- No uniform bound is required for the elevation
- Proof relies critically on the holomorphic coordinates

Soliton resolution for (v) with localized data

Key difficulty: Benjamin-Ono has small solitons, and likely, also (v) !

Conjecture

Any solution to (v) with small localized data resolves into a scattering part and at most one soliton.

Partial results for Benjamin-Ono [Ifrim-T '17] !

Theorem (Ifrim-T '17)

Any solution to Benjamin-Ono with ϵ- small localized data has dispersive decay almost globally in time, i.e. for

$$t \lesssim T_\epsilon = e^{\frac{c}{\epsilon}}$$

- The BO soliton (if any) can only emerge after this time ! (by inverse scattering)
- Work in progress to prove the same result for (v).

Soliton resolution for (b) with localized data

Key difficulty: KdV has small solitons, and also (b) !

Conjecture

Any solution to (b) with small localized data resolves into a scattering part and at most one soliton.

Partial results for KdV [Ifrim-Koch-T '19] !

Theorem (Koch-Ifrim-T '19)

Any solution to KdV with ϵ- small localized data has dispersive decay on quartic time, i.e. for

$$t \lesssim T_\epsilon = \epsilon^{-3}$$

- The KdV soliton (if any) can only emerge after this time ! (by inverse scattering)
- Dispersive shocks can also form at the same time scale.
- Work in progress to prove the same result for (b).

Morawetz inequalities for gravity waves

Theorem (Alazard- Ifrim-T '18)

Let (η, ψ) be a solution for the two dimensional water wave equation (g) or (b) which stays uniformly small in time,

$$\sup_{t \in [0,T]} \|(\eta, \psi)(t)\|_{E_{crit}} \leq \epsilon \ll 1$$

Then the following local energy estimate holds

$$\int_0^T \int_0^1 e(\eta, \psi) dx dt \lesssim \sup_{t \in [0,T]} \|(\eta, \psi)(t)\|_{E^{\frac{1}{4}}}$$

- Result uniform in $T > 0, g > 0, h \geq 1$.
- No prior results that we are aware of.
- Forbids small stationary solutions.
- Result is stated in Eulerian coordinates but proof relies critically on the holomorphic coordinates
- Similar results for gravity/capillary waves at low Bond number.

References

1. Virtual Summer School in Water Waves, MSRI, Aug. 2020, with M. Ifrim, (20 video lectures)

2. Local well-posedness for quasilinear pde's, expository notes, arxiv

3. Testing by wave packets and modified scattering, expository notes, soon to come

3. More to come !

Thank you !