CLAUDIA SCHEIMBAUER: TQFTS AND HIGHER CATEGORIES

We have spent plenty of time this week on objects closely related to topological field theories, including tensor categories and fusion categories. Today, we will begin with topological field theory and see among other things how tensor categories appear out of this formalism.

Definition 0.1. An n-dimensional *topological field theory* (TFT) is a symmetric monoidal functor $Z\colon n\mathcal{C}ob \to \mathcal{V}ect$.

Here $\mathcal{V}ect$ is the symmetric monoidal category of vector spaces with tensor product. $n\mathcal{C}ob$ is a symmetric monoidal category of bordisms:

- The objects of $n\mathcal{C}ob$ are closed, $(n-1)$-dimensional manifolds.[1]
- The morphisms are n-dimensional cobordisms.[2] That is, a morphsim from M to N is a compact n-dimensional manifold X with boundary, together with a diffeomorphism $\partial X \cong M \amalg N$.
- Composition is gluing of cobordisms. Crucially, this must be associative on the nose, which means that we must identify cobordisms which are diffeomorphic rel boundary — so a morphism is in fact a diffeomorphism class of boundary.
- The symmetric monoidal structure is disjoint union of manifolds; the unit is the empty set, which is a compact $(n-1)$-manifold.

If something you care about (e.g. some manifold invariant) is part of a TFT, that means that you can compute it by chopping a manifold into pieces as cobordisms, computing it on those pieces, and then putting them together, at least in principle. This is often a nice thing to have.

Remark 0.2. Why "topological?" Well, you can consider different kinds of *tangential structures* on the manifolds in $n\mathcal{C}ob$, including orientations, spin structures, or principal G-bundles with a flat connection. Most of what we do today will generalize to these settings.

But you can also consider other structures (arbitrary connections, Riemannian metrics, etc.), and studying these functorial field theories as in Definition 0.1 is much harder. ◂

One immediate consequence of the definition of a TFT is that if X is a closed n-dimensional manifold, it defines a cobordism $\varnothing \to \varnothing$, so if Z is a TFT, $Z(M)$ is a map $Z(\varnothing) \to Z(\varnothing)$. Monoidality includes data of an isomorphism $Z(\varnothing) \cong k$, so $Z(X)$ is a linear map $k \to k$, identified with a number. This number, called the *partition function* of X, is a diffeomorphism invariant.

Another lemma one can prove is that for any closed $(n-1)$-manifold M, $Z(M)$ is a finite-dimensional vector space. This is reminiscent of the various finiteness conditions imposed on tensor categories this week, and this is not a coincidence.

Example 0.3. We've already seen a few examples of TFTs this week. In Belaikova's talk, we discussed the Witten-Reshetikhin-Turaev invariants of 3-manifolds associated to a modular tensor category. If this is in addition a spherical fusion category, these specialize to Turaev-Viro invariants, which are computed by a *state-sum construction*: triangulate your manifold, then assign $6j$-symbols to the tetrahedra. ◂

The state-sum construction has an interesting consequence: we've chopped our axiom up not just into codimension-1 submanifolds, but also higher-codimension submanifolds: the edges and vertices. Can we capture this structure in a definition?

Unfortunately, symmetric monoidal categories aren't enough to capture these ideas. One is led to "higher categories," which more naturally incorporate lower-dimensional submanifolds into the cobordism category. We also saw in a previous talk the notion of a modular functor, which assigns invariants not just to 3-dimensional and 2-dimensional manifolds, but also to 1-dimensional manifolds. Here vector spaces are

[1]There are many different variants, e.g. we could ask for manifolds and bordisms to be oriented, or framed, etc., but for now we just want smooth manifolds with no other data.

[2]Well, not quite — but we'll get there in just a moment.

assigned to surfaces, and something new is assigned to 1-manifolds. We can hope this comes from some sort of extended TFT.

Another piece of motivation for extended TFT is: why fusion categories or modular tensor categories? We are led to them when formalizing Turaev-Viro or Witten-Reshetikhin-Turaev invariants, but are there others that we missed? If we understand conceptually why we get these TFTs, that will help us search for new TFTs in other dimensions. And why the specific adjectives that we put in front of fusion categories? Where do they come from?

To mathematically model these extended TFTs, we will add more layers of data. In a TFT as in Definition 0.1, we have a symmetric monoidal category, and we assign things to n- and $(n-1)$-dimensional manifolds.

In an n-dimensional *once-extended TFT*, we replace symmetric monoidal categories with *symmetric monoidal bicategories*, and we can assign data to manifolds of dimension n, $n-1$, and $n-2$. This has been studied by Schommer-Pries [SP09] and Morton [Mor06].

We can further extend to something called a *symmetric monoidal k-category*, which assigns data to manifolds of dimensions from $n-k$ to n. Here we need $k \leq n$.

Another thing you can do is to consider moduli spaces of cobordisms: the idea is that nearby cobordisms should do similar things in a field theory. This leads one to a symmetric monoidal topological category of cobordisms, or alternatively a symmetric monoidal $(\infty, 1)$-category.[3] This assigns invariants to n- and $(n-1)$-categories as usual, but we also have data in all dimensions higher than m, but because that data corresponds to paths or isotopy of paths in a moduli space, those morphisms are all invertible. These have been studied by Tillmann and Madsen, with key input from Galatius-Madsen-Tillmann-Weiss [GTMW09].

Finally, you can combine these into the notion of *fully extended TFT*, where you combine moduli spaces and extended TFT, in a symmetric monoidal (∞, n)-category. We assign things in all dimensions, and above dimension n everything we define is invertible. There are different models of symmetric monoidal (∞, n)-categories and the appropriate generalizations of $n\mathcal{C}ob$, due to work of Lurie [Lur09], Calaque-Scheimbauer [CS19], Verity, and Ayala-Francis-Rozenblyum.

In these various kinds of extended TFT, the cobordism category is some sort of symmetric monoidal higher category, and the target category, generalizing $\mathcal{V}ect$, must also be the same kind of symmetric monoidal higher category. Whatever we put there should allow us to recover the notion of an ordinary TFT from an extended one, in the same way that these cobordism higher categories should contain the data of $n\mathcal{C}ob$.

For example, let $\mathcal{B}ord_{2,1,0}$ denote the symmetric monoidal bicategory of 0-manifolds, 1-manifolds (with boundary), and 2-manifolds (with corners). Consider the morphism category $\mathcal{B}ord_{2,1,0}(\varnothing, \varnothing)$ — its objects are closed 1-manifolds, and its morphisms are (diffeomorphism classes of) compact bordisms between them — so this is just $2\mathcal{C}ob$ again! So if $\mathcal{C}$ is a suitable symmetric monoidal bicategory to use for extended TFT, then in any TFT $Z\colon \mathcal{B}ord_{2,1,0} \to \mathcal{C}$, we can restrict to $\mathcal{B}ord_{2,1,0}(\varnothing, \varnothing)$, i.e. the ordinary bordism category, and Z maps this to $\Omega\mathcal{C} := \mathcal{C}(\mathbf{1}, \mathbf{1})$. Thus we need $\Omega\mathcal{C} \simeq \mathcal{V}ect$.

Example 0.4. Here are two symmetric monoidal bicategories $\mathcal{C}$ with $\Omega\mathcal{C} \simeq \mathcal{V}ect$ as desired, and which are useful targets for once-extended TFT.

- The *Morita bicategory* $\mathcal{A}lg_1(\mathcal{V}ect)$ is the bicategory whose objects are algebras over k, whose 1-morphisms $A \to B$ are (B, A)-bimodules, and whose 2-morphsims are bimodule homomorphisms. $\Omega\mathcal{A}lg_1(\mathcal{V}ect)$ is the symmetric monoidal category of (k, k)-bimodules, which is the same thing as vector spaces.
- The *bicategory of k-linear categories* $\mathcal{L}in\mathcal{C}at_k$ has for its objects the (finite, or finitely presented) k-linear categories, for 1-morphisms the k-linear functors, and for 2-morphims the k-linear natural transformations. The unit is $\mathcal{V}ect$, and a k-linear functor $\mathcal{V}ect \to \mathcal{V}ect$ determines and is determined by where it sends k, which can be an arbitrary vector space; hence $\Omega\mathcal{L}in\mathcal{C}at_k \simeq \mathcal{V}ect$. ◀

Example 0.5. For another example, $\mathcal{A}lg_1(\mathcal{L}in\mathcal{C}at_k)$ is a symmetric monoidal $(\infty, 3)$-category, and is a natural target for three-dimensional TFT, and is useful for understanding Turaev-Viro theory. The objectds are monoidal k-linear categories (which includes tensor categories); the 1-morphisms are k-linear bimodule categories; the 2-morphisms are k-linear bimodule functors; and the 3-morphisms are k-linear bimodule natural transformations. ◀

[3] These are sometimes just called symmetric monoidal ∞-categories.

Ok, what about higher n? In general, we can take some category $\mathcal{A}lg_n(\mathcal{C})$ of higher algebras of some sort. Calaque-Scheimbauer [CS19] do this geometrically, and Haugseng [Hau17] does this with ∞-operads. This can be extended further in the spirit of Example 0.4, as done by Johnson-Freyd and Scheimbauer [JFS17].

Definition 0.6. A *fully extended TFT* is a symmetric monoidal functor $Z\colon \mathcal{B}ord_{n,\dots,0} \to \mathcal{C}$, where $\mathcal{C}$ is some symmetric monoidal (∞, n)-category.

Theorem 0.7 (Cobordism hypothesis)**.** *The map $Z \mapsto Z(\mathrm{pt})$ defines an equivalence from the symmetric monoidal (∞, n)-category of fully extended TFTs valued in $\mathcal{C}$ to the symmetric monoidal (∞, n)-groupoid of units of n-dualizable objects in $\mathcal{C}$.*

This theorem was conjectured by Baez-Dolan [BD95], and proven in dimension 2 by Schommer-Pries [SP09], Pstrągowski [Pst14], and Hopkins-Lurie. In general it's sketched by Lurie [Lur09] and Ayala-Francis [AF17], with closely related work by Harpaz and Schommer-Pries.

This leads to a strategy for finding fully extended TFTs: first, find an interesting symmetric monoidal (∞, n)-category $\mathcal{C}$, e.g. via $\mathcal{A}lg_n(\mathcal{C})$ as above; then, find n-dualizable objects in $\mathcal{C}$.

Example 0.8. Douglas, Schommer-Pries, and Snyder [DSPS13] studied this for the 3-category of tensor categories inside $\mathcal{A}lg_1(\mathcal{L}in\mathcal{C}at_k)$, provided k is algebraically closed of characteristic zero. In particular, they showed that fusion categories are 3-dualizable, leading to fully extended 3d TFTs. (To get Turaev-Viro, you need to do something else to descend from framings to orientations.)[4] ◀

Here are some other related results.

- Brandenberg-Chirvasitu-Johnson-Freyd [BCJF15] show that $\mathcal{L}in\mathcal{C}at_k$, having enough compact projectives implies 2-dualizability.
- Brochier-Jordan-Snyder [BJS18] showed that a fusion category in $\mathcal{A}lg_1(\mathcal{L}in\mathcal{C}at)_k$ is 3-dualizable, and that a fusion-category in $\mathcal{A}lg_2(\mathcal{L}in\mathcal{C}at)_k$ (the 4-category of braided k-linear categories) is 4-dualizable, so we get 4-dimensional framed TFTs.
- Gwilliam-Scheimbauer [GS18] showed that in $\mathcal{A}lg_n(\mathcal{C})$, under mild assumptions on $\mathcal{C}$, *all* objects are dualizable.
- Douglas-Reutter [DR18] conjecture that fusion 2-categories are 4-dualizable in $\mathcal{A}lg_1(2\mathcal{C}at_k)$.

So we understand some things and are still learning others. Here are a few things we are still working on.

- Well, actually working with these TFTs, rather than knowing they're defined, is not necessarily made any easier by the cobordism hypothesis. Working with the TFTs that the work above has shown exists is still largely open.
- The target (∞, n)-categories considered are not the only ones we might be interested in. What does the cobordism hypothesis say for other categories?
- Some field theories aren't fully extended, and we'd have to modify this framework. For example, relative field theories, related to boundary conditions, are studied by Stolz-Teichner and Freed-Teleman [FT14].
- If you're only k-dualizable for $k < n$, then you can build a "categorified TFT" which assigns invariants up to dimension k, but not farther. Less is known about these theories.
- What about other tangential structures? We know that in dimension 3, oriented theories are related to pivotal and spherical structures, by work of Douglas, Schommer-Pries, and Snyder [DSPS13]. But pretty much everything else here is open.

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