VICTOR OSTRIK: INTRODUCTION TO FUSION CATEGORIES, II

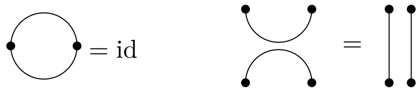
We begin by discussing diagrammatics for $\mathcal{V}ec^{\omega}_{\mathbb{Z}/n}$, following work of Agustina Czenky.

The objects of $\mathcal{V}ec^{\omega}_{\mathbb{Z}/n}$ are all tensor products of $1 \in \mathbb{Z}/n$, which in diagrammatics is denoted $\uparrow$; thus, i is denoted $\uparrow\uparrow \overset{i}{\cdots} \uparrow$. To simplify notation, temporarily let $n = 2$. The generating morphisms are

(0.1)

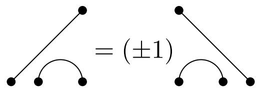
and there are some relations:

(0.2)



and

(0.3)



where the sign is $+1$ for $[\omega] = 0$ in $H^3(B\mathbb{Z}/2; \mathbb{R}/\mathbb{Z})$ and -1 for $[\omega]$ nonzero. For more general n, the factor is some n^{th} root of unity ζ determined by ω.[1]

This generators-and-relations description of $\mathcal{V}ec^{\omega}_{\mathbb{Z}/n}$ allows us to uncover a universal property.

Proposition 0.4. *Let $\mathcal{C}$ be a fusion category. Then there is a natural bijection between tensor functors $F\colon \mathcal{V}ec^{\omega}_{\mathbb{Z}/n} \to \mathcal{C}$ and isomorphism classes of data of an object $X \in \mathcal{C}$ and an isomorphism $\theta\colon X^{\otimes n} \xrightarrow{\cong} \mathbf{1}$.*

The idea is that, looking at the diagrammatics of $X := F(1)$, we have two different isomorphisms $X^{\otimes(n+1)} \xrightarrow{\cong} X$, and one must be ζ times the other.

The slightly more sophisticated way to say this is that functors $\mathcal{V}ec^{\omega}_{\mathbb{Z}/n} \to \mathcal{C}$ form a category, and the data (X, ϕ) as above forms a category, and the above bijection can be promoted to an equivalence of categories.

Example 0.5. Assume n is odd, so that we can use $2 \in \mathbb{Z}/n$ as a generator. This gives two diagrams that you can compare, and one is a multiple of another. It turns out the factor is ζ^4, and this gives the action of $\mathrm{Aut}(\mathbb{Z}/n)$ on $H^3(\mathbb{Z}/n; k^\times)$ from last time. ◂

Now let us use this to study tensor functors $F\colon \mathcal{V}ec^{\omega}_{\mathbb{Z}/n} \to \mathcal{B}imod_R$, where R is a k-algebra. These are classified by (R,R)-bimodules X together with an isomorphism $X^{\otimes n} \xrightarrow{\cong} R$; in particular, X is invertible. Hence we can restrict our search to $\mathrm{Pic}(R) \subset \mathcal{B}imod_R$, the subcategory of invertible bimodules. Inside $\mathrm{Pic}(R)$, we also have $\mathrm{Out}(R)$; as described last time, an outer automorphism defines an (isomorphism class of) (R,R)-bimodules.

Exercise 0.6. Assuming $Z(R) \cong k$, let $\theta \in \mathrm{Out}(R)$. Show that $\theta(g) = \zeta g$ for $g \in R^\times$ and some root of unity ζ. If this is too difficult at first, take a look at some examples; see if you can give an example for any ζ.

One big open problem in this field is to classify all fusion categories. This is of course way too hard, given that it's more difficult than the classification of finite groups, but as with the classification of finite groups, intermediate results are interesting, possible, and useful.

Theorem 0.7 (Ocneanu rigidity (Etingof-Nikshych-Ostrik [ENO05]))**.** *Fusion categories over $\mathbb{C}$ have no deformations.*

This was originally conjectured by Ocneanu. We won't say precisely what a deformation of a fusion category is, but the data of associativity in a fusion category is matrices satisfying some equations, modulo the action of some symmetry group. Ocneanu rigidity amounts to there being only finitely many orbits of solutions under this group action.

[1] Conversely, any choice of ζ is given by some ω, though it's not always easy to work this out in practice.

Corollary 0.8. *There are countably many tensor equivalence classes of fusion categories over $\mathbb{C}$.*

For example, $\mathcal{V}ec_G^\omega$ is classified by $H^3(G;\mathbb{C}^\times)$, which is a finite group.

Remark 0.9. Ocneanu rigidity is open in positive characteristic; the speaker expects it to be true, but for different reasons. The proof in characteristic zero uses a tool called *Davydov-Yetter cohomology* for fusion categories — this vanishes in characteristic zero, which implies Theorem 0.7, but is known to not vanish in characteristic p in general. ◀

Example 0.10. Another rich source of interesting examples of fusion categories are quantum groups at roots of unity. The construction is quite complicated. For example, fix an integer λ, called the *level*; then, one can build a fusion category $\mathcal{C}(\mathfrak{sl}_2,\lambda)$ as follows. There are $\lambda+1$ simple objects $L_0,\dots,L_\lambda$, and the fusion rules are determined by

$$L_i\otimes L_1 = L_1\otimes L_i = L_{i-1}\oplus L_{i+1}, \tag{0.11}$$

where $L_{-1}=L_{\lambda+1}=0$; this suffices to determine the rest of the fusion rules. This looks reminiscent of the representation theory of $\mathfrak{sl}_2$, but "cut off" at λ. ◀

Any fusion category not equivalent to $\mathcal{V}ec_G^\omega$ or a quantum group is called *exotic*. Examples of exotic fusion categories are constructed using subfactor theory; many are related to something called *near group categories*. Such a category has as its set of simple objects $G\amalg\{X\}$, where G is a finite group. The tensor product on elements of G is just multiplication, and the remaining rules are

$$g\otimes X = X\otimes g = X \tag{0.12a}$$

$$X\otimes X = \bigoplus_{g\in G} g\oplus nX, \tag{0.12b}$$

for some $n\in\mathbb{N}$.

It's not necessarily true that one can find an associator compatible with this data, but often one can. For $n=0$, these are *Tambara-Yamagami categories*, which are relatively well-studied. For $n>0$, Evans-Gannon [EG14] showed that either $n=\#G-1$ or $\#G\mid n$. Moreover, if $n\geq\#G$, then G must be abelian, which is a nice simplification! Izumi-Tucker [IT19] considered the cases where $n=\#G-1$ and $n=\#G$ — in the latter case, there are only finitely many examples, and for $n=2\#G$, there's a single example with $G=\mathbb{Z}/3$. In general, it's open whether there are a finite number of examples for a fixed n as a function of $\#G$.

There are also various useful constructions which, given some fusion categories, produce more. One example is Deligne's tensor product $\mathcal{C}\boxtimes\mathcal{D}$ of fusion categories, which is again a fusion category.

Definition 0.13. An *associative algebra* in a fusion category $\mathcal{C}$ is an object $A\in\mathcal{C}$ together with $\mathcal{C}$-morphisms $m\colon A\otimes A\to A$ and a unit $i\colon\mathbf{1}\to A$, subject to axioms guaranteeing associativity of m and that i is a unit for m.

Given an associative algebra A in $\mathcal{C}$, we can define A-module and (A,A)-bimodule objects in $\mathcal{C}$, analogous to the usual case.

Exercise 0.14. Write these definitions down.

The upshot is, given an associative algebra A in $\mathcal{C}$, the category of (A,A)-bimodule objects, denoted ${}_A\mathcal{C}_A$, is a tensor category. The monoidal product is $\otimes_A$, and the unit is A. This is generally not a fusion category — and indeed, even if A is just an algebra in vector spaces, its category of bimodules generally isn't fusion! But if A satisfies an assumption called *separability*, then ${}_A\mathcal{C}_A$ is semisimple and rigid, which is pretty close to being fusion — all we need is that $\mathbf{1}\in{}_A\mathcal{C}_A$ is indecomposable. A sufficient (but not necessary) condition for this is $\mathrm{Hom}_{\mathcal{C}}(\mathbf{1},A)\cong k$.

Remark 0.15. If we start with $\mathcal{C}=\mathcal{V}ec_G^\omega$ and look for bimodule categories for algebras in $\mathcal{C}$, the fusion categories we obtain are called *group-theoretical fusion categories.* ◀

Another procedure to obtain fusion categories is called *graded extensions*, building a G-graded fusion category, where G is a finite group, out of an ungraded fusion category.

Here are two extremely useful tools for classifying fusion categories.

(1) The Drinfeld center of a fusion category is a modular tensor category. Modular tensor categories have a lot of structure, and this is helpful for learning about fusion categories.
(2) Diagrammatic methods are helpful, as we saw at the beginning of today's lecture.

References

[EG14] David E. Evans and Terry Gannon. Near-group fusion categories and their doubles. *Adv. Math.*, 255:586–640, 2014. https://arxiv.org/abs/1208.1500. 2

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[IT19] Masaki Izumi and Henry Tucker. Algebraic realization of noncommutative near-group fusion categories. 2019. https://arxiv.org/abs/1908.01655. 2