1)

$$\zeta(\lambda) = \prod_{k=0}^{n-1} D_k(\lambda)^{(-1)^k}$$

$$\log D_k^{\prime}(\lambda) = \int_{0}^{\infty} \operatorname{tr} e^{-tP_k} e^{it\lambda}\, dt \qquad A = B - C$$

trace formula

$$\overset{\text{formally}}{=} \operatorname{tr}^{\flat} \underbrace{P_k (P-\lambda)^{-1}}_{\text{understand this}}$$

$$\boxed{P_k = \mathcal{L}_V \ldots \qquad P^1 \ldots \qquad i_V \omega = 0}$$

$\operatorname{tr}^{\flat} A$ ok if $\mathrm{WF}(K_A) \cap N^* \Delta = \emptyset$

1st task $\quad P - \lambda : D_{sG} \longrightarrow H_{sG}$ Fredholm, $\operatorname{Im} \lambda > -\frac{s}{C}$

$$D_{sG} := \{ u \in \mathcal{D}' : u \in H_{sG},\ Pu \in H_{sG} \}$$

$$H_{sG} := e^{-sG} L^2, \quad G(\xi) = m_G(x,\xi) \log |\xi|$$

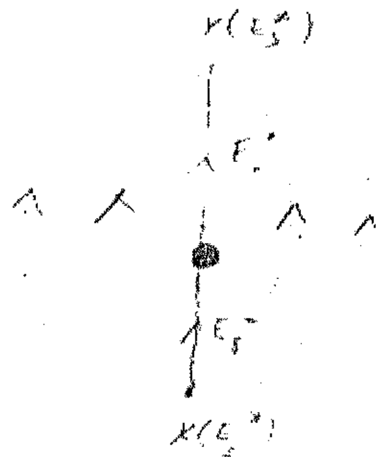
Invert after adding a compact perturbation

$\chi(\ldots) \in \Psi^{0+}$ $\quad G(\chi) = m_G(x,\xi) \log |\xi| \, (1 - \chi(x,\xi))$

$\chi = 1$ $\qquad H_{sG(\chi)} = e^{-sG(\chi)} L^2$

$H_{sG(\chi)} = H_{sG}$ with norms depending on χ

P.1)



trapped set at G

cusp zero section

gain compactness — move from higher regularity to lower

$H^s \hookrightarrow H^t$ compact $s > t$

$m_G(x,\xi) \equiv 1$ near $\kappa(E_s^*)$ $\boxed{H_p G \le 0}$

$m_G(x,\xi) \equiv -1$ —"— $\kappa(E_u^*)$

$H_{sG} \cong H^s$ microl. near $\kappa(E_s^*)$

$\cong H^{-s}$ —"— $\kappa(E_u^*)$

P on $H^{sG} \rightsquigarrow e^{sG} P e^{-sG} = P + i\,Op(H_p G)$

$P + s[G,P] + \Psi^{-\infty} + \Psi^{-2+1}(M)$

2) $P_\delta(z) = hP - iQ_\delta - z$ ← we damped the $z = O(h)$ "trapped set" $\boxed{7-36}$

Q_δ microlocalised to $|\xi| \le 2\delta$, semiclassical $Q_\delta \in \Psi_h^0(M)$

$Q_\delta = 1$ on $|\xi| \le \delta$

Theorem $P_\delta(z) : H_{sc}(h) \longrightarrow H_{sc}(h)$, $-\frac{\delta}{C} h \le \operatorname{Im} z \le 1$, $|\operatorname{Re} z| \le h^\epsilon$

is invertible

$\| P_\delta(z)^{-1} \|_{H_{sc} \to H_{sc}} \le C h^{-1}$ &

$WF_h'(P_\delta(z)^{-1}) \cap T^*(X \times X) \subset \Delta(T^*X) \cup \Omega_+$

$\Omega_+ := \{ (e^{tH_p}(x,\xi), (x,\xi)) : t \ge 0, \; p(x,\xi) = 0 \}$

→ this gives the Fredholm property!

$P_\delta(z) u = f$ Estimate Au, $A \in \Psi_h^0$ in terms of Bf, $B \in \Psi_h$

$\Longrightarrow \|u\|_{H_{sc}(h)} \le h^{-1} \|f\|_{H_{sc}(h)}$

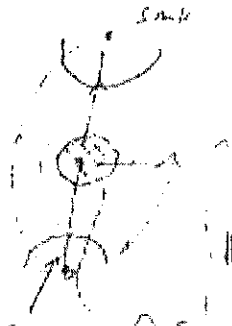
& will get control of $WF_h(u)$ in terms of $WF_h(f)$ which provides an estimate on $WF_h'(P_\delta(z)^{-1})$

$\left((y,\eta,x,\xi) \in T^*(X \times X) \setminus (\Delta(T^*X) \cup \Omega_+) \right)$

$\exists$ U = nbhd (x,ξ), V = nbhd (y,η) s.t.

$\forall u \in H_{sc}$, ~~$WF_h(u)$~~ $P_\delta(z)u = f$, $WF_h(f) \subset U$, $\Longrightarrow WF_h(u) \cap V = \emptyset$

3]



discuss $\kappa(\Sigma_s^{\pm})$, $\kappa(\Sigma_u^{\pm})$
highly singular

$\in C^{\infty}$ ...

$\cdot H_{sG}$

Case 0 $WF_h(A) \cap WF_h(I-Q) = \emptyset$

$\|Au\|_{H^s} = \|A^s e^{sG} u\| \le C\|B_1 P_{\delta,s}(z)u\| + O(h^{\infty})$

$Q\ \delta$

$A_s = e^{sG} A e^{-sG}$

Case 1 Source $s \geq 0$

$\|Au\|_{H^s}$ ... $\Longrightarrow \|Au\|_{H^s_{sG}} \le C\|B_1 f\|_{H_{sG}} + O(h^{\infty})\|u\|$

$\le Ch^{-1}\|B_1 f\|_{H^s}$

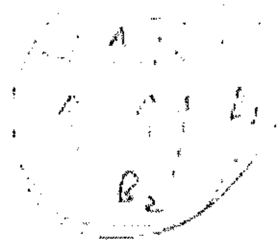
$+ O(h^{\infty})$ radial source high regularity

Case 2 A "in the middle" regular propagation

$\|Au\|_{H_{sG}} = \|A^s e^{sG} u\|_{L^2} \le$

$\le \frac{1}{h}\|B_1 f\|_{H_{sG}} + \|B_2 u\|_{H^{sG}} + O(h^{\infty})$

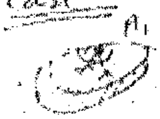
$\le \frac{1}{h}\|B_1 f\|_{H^s_{sG}} + \|B_2 f\|_{H_{sG}}$



micro. near

$e^{-TH_p}(WF_h(A))$

Case 3 Sink $H_{sG} \simeq H^{-s}$ $s \gg 1$ $+ O(h^{\infty})$



$\|Au\|_{H_{sG}} \le \frac{1}{h}\|B_1 f\|_{H_{sG}} + \|B_2 u\|_{H^{-sG}}$

$\|_{H^{-s}}$ H^{-s} $WF_h(B_2) \cap \kappa(E_s^{*}) = \emptyset$

2.4) $\|P_2 u\|_{H^{s}}$ can be controlled by propagation estimates

$\Longrightarrow$ $\|u\|_{H^{s}} \le \frac{1}{h}\|f\|_{H^{s}} + O(h^\infty)\|u\|_{H^{s}}$

$\Longrightarrow$ desired estimate.

Recall Theorem: Suppose $\operatorname{Im}\lambda_0 > -\frac{1}{C}$, then near λ_0

$$(P-\lambda)^{-1} = R_{hol}(\lambda) + \sum_{1}^{J} \frac{(P-\lambda_0)^{j-1}\Pi}{(\lambda-\lambda_0)^{j}}$$

$$WF'(R_{hol}(\lambda)) \subset \Delta(T^*X) \cup \Omega_+ \cup (E_u^* \times E_s^*)$$

$$WF'(\Pi) \subset E_u^* \times E_s^*$$

$WF_h'(\ ) \cap T^*(X\times X) \subset \Delta(T^*X) \cup \Omega_+$

$$R(\lambda) = h\left(R_\delta(z) - i R_\delta(z)\, Q_\delta\, R_\delta(z) - R_\delta(z)\, Q_\delta\, R(z)\, Q_\delta\, R_\delta(z) \right)$$

multiple to $(\lambda-\lambda_0)^{J}$

$WF_h'(\ ) \subset \{|\xi| \le 2\delta\}$

$$WF_h'(\ ) \cap T^*(X\times X) \subset \Upsilon_\delta$$

$$\Upsilon_\delta = \{(\rho', \rho) : \exists t, s \ge 1, |\xi(e^{tH_p}(\rho))| \le \delta, |\xi(e^{-sH_p}(\rho'))| \le \delta\}$$

$$\bigcap_{\delta \to 0} \Upsilon_\delta = E_u^* \times E_s^*$$

5)

$$\partial_\lambda \ln \zeta_1(\lambda) = \frac{1}{i} e \int_0^\infty e^{i\lambda t} e^{-i\lambda_0 t}\, \mathrm{tr}^b\, \varphi_{-t_0}\, e^{-itP}\, dt$$

WF cond. is ABF

go via b-str

$$= \frac{1}{i} \lim_{\varsigma \to 0} \int_0 e^{-i\lambda_0 t_0} e^{i\lambda t}\, \mathrm{tr}\; E_\varsigma \varphi_{-t_0} e^{-itP} E_\varsigma \, dt$$

$$= \frac{1}{i} \lim_{\varsigma\to 0} e^{-i\lambda_0 t}\, \mathrm{tr} \int_0 e^{i\lambda t} E_\varsigma \varphi_{-t_0} e^{-itP} E_\varsigma \, dt \qquad E_\varsigma \in \Psi^{-\infty}, \quad E_\varsigma \to I + \Psi^{0+}$$

$$= - \lim_{\varsigma \to 0} e^{-i\lambda_0 t}\, \mathrm{tr}\; E_\varsigma e^{-it_0 P} R(\lambda) E_\varsigma$$

WF cond. is there

$$= - e^{-i\lambda_0 t}\, \mathrm{tr}^b \left(\varphi^*_{-t_0} R(\lambda) \right)$$

On the singular part $\sum_0^J \frac{(P-\lambda_0)^{j-1}\Pi}{(\lambda-\lambda_0)^j}$

tr^b is the regular tr (on H_{SG})

So residue is $\mathrm{tr}_{H_{SG}} \Pi \in \mathbb{N}_0$