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NOTETAKER CHECKLIST FORM

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Name: Ian Coley **Email/Phone:** msri@iancoley.org

Speaker's Name: Gijs Heuts

Talk Title: Spectra Lie algebras and unstable homotopy theory

Date: 3 / 25 / 19 **Time:** 3 : 30 am / [✓ pm] (circle one)

Please summarize the lecture in 5 or fewer sentences:
This talk defines in a new way the Lie operad in spectra. Using it, they prove a Quillen-type theorem identifying certain homotopy types of spectra with Lie algebras, but no longer confined to characteristic zero.

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(This is **NOT** optional, we will **not pay** for **incomplete** forms)

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SPECTRAL LIE ALGEBRAS AND UNSTABLE HOMOTOPY THEORY

GIJS HEUTS

Outline:

Part 1, what are spectra Lie algebras? They should be Lie algebras in spectra such that if we look over $\mathbb{Q}$ or $H\mathbb{Q}$ we should get differential graded (dg) Lie algebras over $\mathbb{Q}$. Why use them?

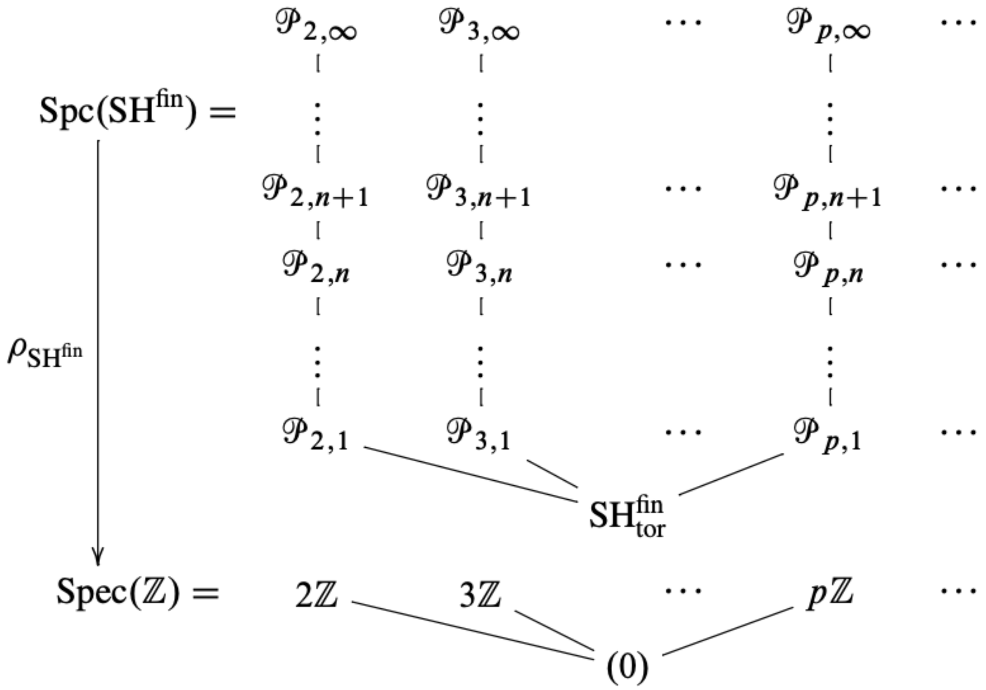
- Formal deformation theory if a dg Lie algebra over $\mathbb{Q}$. Brantner-Mathew use (spectral) partition Lie algebras to do formal deformation theory in positive charactersitic
- Quillen's description of rational homotopy theory. The following have the same homotopy theories:

$$\begin{aligned}\{\text{connected dg Lie algebras over } \mathbb{Q}\} &\simeq \{\text{simply connected rational spaces}\} \\ &\simeq \{\text{simply connected cocommutative dg coalgebras over } \mathbb{Q}\}\end{aligned}$$

Part 2, generalize Quillen's results to "v_n-periodic localizations" of $\mathcal{S}_*$ the homotopy theory of pointed spaces. Recall that $\operatorname{Spec}\mathbb{S}$, i.e. the Balmer spectrum of finite spectra, is considerably larger than $\operatorname{Spec}\mathbb{Z}$ [1]

Notes by Ian Coley.

[1] Paul Balmer, *Spectra, Spectra, Spectra*, https://projecteuclid.org/euclid.agt/1513715144



For each prime $p \in \mathbb{Z}$, we have a corresponding twoer which we call $K(n)_p$ (written $\mathscr{P}_{p,n}$ above) for $n \leq 1$. There is a corresponding localization of $\mathcal{S}_*$ for each of these as well which we write $\mathcal{S}_{v_n}$ (where we will be fixing p so do not include it in the notaiton). Rational corresponds to $n = 0$, at which all the different $K(n)_p$ coincide.

Theorem 1. The v_n-periodic localization $\mathcal{S}_{v_n}$ is equivalent to $\mathrm{Lie}(\mathrm{Sp}_{T(n)})$, some category of spectral Lie algebras.

I. Spectral Lie algebras

Consider the category of augmented E_∞-ring spectra $\mathbf{CAlg}^{\mathrm{aug}}(\mathrm{Sp})$. There is a trivial functor Sp to this category which assigns to $X \in \mathrm{Sp}$ the square-zero extension $\mathrm{triv}(X) = \mathbb{S} \oplus X$. This functor preserves limits, so by the adjoint functor theorem it admits a left adjoint which we will call TAQ and can be thought of as the cotangent fibre, or topological Andre-Quillen homology (hence the name), or derived indecomposables. Specifically,

$$TAQ(A \to \mathbb{S}) = \mathbb{L}_A \otimes A\mathbb{S}$$

where $\mathbb{L}_A$ is the cotangent complex.

Facts: first,

$$TAQ(\mathrm{triv}(X)) \cong \bigoplus_{k\geq 1} \left(\Pi_k \otimes X^{\otimes k}\right)_{h\Sigma_k}$$

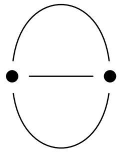
where Π_k is the kth partition complex (more on that soon). Second, $TAQ \circ \mathrm{triv}$ is a comonad, which means that the Π_k assemble into a cooperad, so dualizing gives an operad which we call $\mathbb{L}$ the *spectral Lie operad*, first constructed by Ching and Salvatore independently.

Definition 2. For $k \geq 2$, define $\mathrm{Part}^{\pm}(k)$ to be the poset of partitions of $\{1, \ldots, k\}$ which are proper and nontrivial, and let $\Pi_k = \Sigma|\mathrm{Part}^{\pm}(k)|^{\diamond}$, where $\diamond$ denotes the unreduced suspension. Let $\Pi_0 = \Pi_1 = *$.

Remark 3. Disregarding the natural Σ_k-action, $\Pi_k \simeq \bigvee_{(k-1)!} S^{k-1}$

Example 4.

- $\mathrm{Part}^{\pm}(2) = \varnothing$, so $\Pi_2 = S^1$.
- $\mathrm{Part}^{\pm}(3) = \{*, *, *\}$ three disjoint points, so

$\Pi_3 \simeq \Sigma\,($  $) = S^2 \vee S^2$

- $\mathrm{Part}^{\pm}(4)$ is also possible to draw, but harder convince yourself of.

Facts about $\mathbb{L}$:

(1) The homology of $\mathbb{L}$ gives us the Lie operad in **Ab** with a degree shift:

$$\begin{aligned} H_*\mathbb{L}(n) &\cong \mathrm{Lie}(n)[1-n] \text{ twisted by the sign representation of } \Sigma_n \\ &\cong \mathrm{Lie}(n)[1] \otimes (\mathbb{Z}[-1])^{\otimes n} \end{aligned}$$

In particular, if M is a graded abelian group, a $H_*\mathbb{L}$-algebra structure on M is the same thing as a graded Lie algebra structure on $M[-1]$, i.e. degree shift is no big deal.

(2) $\mathbb{L}$ is isomorphic to the cobar complex on the Comm cooperad, and by general machinery this automatically makes it an operad.

II. v_n-periodic unstable homotopy theory

Fix a prime p and work everywhere henceforth p-locally, giving us only a tower of $K(n)$ in $\mathrm{Spec}(\mathbb{S}_p^\wedge)$ to think about. They correspond to thick subcategories of finite p-local spectra of type $\geq n$ which we denote

$$\mathrm{Sp}^{\mathrm{fin}}_{(p)} \supset \mathrm{Sp}^{\mathrm{fin}}_{\geq 1} \supset \cdots \supset \mathrm{Sp}^{\mathrm{fin}}_{\geq n} \supset \cdots$$

Definition 5. A spectrum X is *of type* n if $K(m)_* X = 0$ for $m < n$ and is nonzero if $m = n$. A space X is of type n if $\Sigma^\infty_+ X$ is of type n.

Definition 6. A v_n*-self map* of X is $v\colon \Sigma^d X \to X$ (for some d) such that $K(m)_* v$ is an isomorphism if $m = n$ and is nilpotent otherwise.

Example 7. S^k for $k \geq 1$ is of type 0, and $p\colon S^k \to S^k$ is a v_0-self map. The quotient S^k/p (i.e. the mod p Moore space) is type 1, and the Adams map (for odd p) $\alpha\colon \Sigma^{2(p-1)} S^k/p \to S^k/p$ is a v_1-self map. Continuing on, the cofibre of α is type 2,. . .

Theorem 8 (Mitchell)**.** Finite type n spaces exist for all $n \in \mathbb{N}$.

Theorem 9 (Hopkins-Smith)**.** If X is a finite space of type n, then some $\Sigma^d X$ has a v_n-self map. Moreover, they are asymptotically unique.

v_n-periodic homotopy groups

Let X be a pointed space, V a type n space with $v\colon \Sigma^d V \to V$ a v_n-map. Then define

$$v_n^{-1}\pi_*(X, V) = v^{-1}\pi_*(\mathrm{Map}_*(X, V))$$

This turns out not to depend on the choice of v (asymptotically).

Definition 10. A map $f\colon X \to Y$ of spaces/spectra is a v_n*-periodic local equivalence* if and only if $v_n^{-1}\pi_*(f, V)$ is an isomorphism (and this actually doesn't depend on V). Let $\mathcal{S}_{v_n}$ be the localization of pointed spaces $\mathcal{S}_*$ at the v_n-local equivalences (not stable, so use ∞-categorical techniques). Let $\mathrm{Sp}_{T(n)}$ be the localization of Sp at the same.

Remark 11. $\mathrm{Sp}_{T(n)}$ is equivalent to the Ind-completion of the Verdier quotient $\mathrm{Sp}^{\mathrm{fin}}_{\geq n} / \mathrm{Sp}^{\mathrm{fin}}_{\geq n+1}$.

The v_n-periodic homotopy groups of X are homotopy groups of an associated spectrum

$$\Phi_V(X) = \mathrm{colim}(\Sigma^\infty \mathrm{Map}_*(V, X) \xrightarrow{v^*} \Sigma^{\infty - d} \mathrm{Map}_*(V, X) \to \cdots)$$

which is called the *telescoptic functor* $\Phi_V\colon \mathcal{S}_* \to \mathrm{Sp}_{T(n)}$, factoring through $\mathcal{S}_{v_n}$. It doesn't really depend on V, so let's lose it from the notation.

Definition 12. The Bousfield-Kuhn functor is a functor $\Phi\colon \mathcal{S}_{v_n} \to \mathrm{Sp}_{T(n)}$ defined by

(1) $\Phi_V \cong \underline{\mathrm{Map}}(V, \Phi)$ as spectra (so the righthand side is the mapping spectrum, not just the space)
(2) $\Phi\Omega^\infty \cong L_{T(n)}\colon \mathrm{Sp} \to \mathrm{Sp}_{T(n)}$ the localization functor, so the localization from spectra to type n spectra factors through infinite loop spaces.

Theorem 13 (Bousfield)**.** Φ has a left adjoint $\Theta\colon \mathrm{Sp}_{T(n)} \to \mathrm{Sp}$.

Theorem 14 (Eldred-H-Mathew-Meier)**.** Φ is a monadic functor, so $\mathcal{S}_{v_n}$ is isomorhpic to algebras in $\mathrm{Sp}_{T(n)}$ over the monad $\Phi\Theta$. We have two adjunctions:

$$\begin{tikzcd}
\mathrm{Sp}_{T(n)} \arrow[d, bend left=60, "\Omega^\infty"] \\
\mathcal{S}_{v_n} \arrow[u, bend left=60, "\Sigma^\infty_{v_n}"] \arrow[d, bend left=60, "\Phi"] \\
\mathrm{Sp}_{T(n)} \arrow[u, bend left=60, "\Theta"]
\end{tikzcd}$$

and we know that $\Phi\Omega^\infty \simeq \mathrm{id}$, so we can conclude that $\Sigma^\infty_{v_n}\Theta \simeq \mathrm{id}$ too. This is analogous to the following for an operad $\mathcal{O}$ with $\mathcal{O}(0) = 0$ and $\mathcal{C}$ a stable category

$$\begin{tikzcd}
\mathcal{C} \arrow[d, bend left=60, "\mathrm{triv}"] \\
\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \arrow[u, bend left=60, "``TAQ''"] \arrow[d, bend left=60, "U"] \\
\mathcal{C} \arrow[u, bend left=60, "F"]
\end{tikzcd}$$

where $U \circ \mathrm{triv} = \mathrm{id}$ and the other composite is also the identity.

There's also a coalgebra version with the same conclusion.

Theorem 15. $\Phi\Theta$ is the free spectral Lie algebra monad,

$$\Phi\Theta(X) \cong \left[\bigoplus_{k\geq 1} (\mathbb{L}(k) \otimes X^{\otimes k})_{h\Sigma_k}\right]_{L_{T(n)} \text{ localized}}$$

Remark 16. We've done part of Quillen's story, but what about the cocommutative coalgebras? Unfortunately,

$$\mathcal{S}_{v_n} \overset{\Sigma^\infty_{v_n}}{\to} \mathrm{coCAlg}(\mathrm{Sp}_{T(n)}$$

is *not* an equivalence, and neither are the categories abstractly, unlike the rational case.

Nonetheless, there is a comparison map

$$\Phi(X) \to \underline{\mathrm{Map}}(TAQ(\mathbb{S}^{X_+}), \mathbb{S}_{T(n)})$$

which generally isn't an equivalence (for the same reason) but is sometimes, e.g. $X = S^k$ [Behrens-Rezk, Arone-Mahowald].

Ingredients of the proof: Say $F\colon \mathrm{Sp}_{T(n)} \to \mathrm{Sp}_{T(n)}$ is *coanalytic* if it is of the form

$$F(X) = \bigoplus_{k\geq 1} (\mathcal{O}(k) \otimes X^{\otimes k})_{h\Sigma_k}$$

for some symmetric sequence $\mathcal{O}$.

Theorem 17. The obvious map from symmetric sequences to coanalytic functions on $\mathrm{Sp}_{T(n)}$ is an equvalence, so maps between functors must arise from maps of symmetric sequences. This is pretty unique to the $T(n)$-local setting.

This theorem is proved using Tate vanishing (Kuhn) and a nilpotence trick (Mathew).

As a consequence, a (co)operad in $\mathrm{Sp}_{T(n)}$ is the same thing as a coanalytic (co)monad.

Proposition 18 (with Lurie)**.** $F\colon \mathrm{Sp}_{T(n)} \to \mathrm{Sp}_{T(n)}$ is coanalytic if and only if F preserves sifted colimits.

This is true rationally, but also true in this case.

Corollary 19. $\Phi\Theta$ preserves sifted colimits, so it corresponds to some operad in symmetric sequences.

and that lets one finish the proof.