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# NOTETAKER CHECKLIST FORM

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Name: ORI KATZ Email/Phone: ORIKATZ.OK@gmail.com

Speaker's Name: Tudor S. Ratiu

Talk Title: Momentum map for automorphism groups

Date: 10/11/18 Time: 9:30 am / pm (circle one)

Please summarize the lecture in 5 or fewer sentences: Raitu presents an enlarged definition of the momentum map with values in groups, in order to include conservation laws that are discrete in nature, such as topological information. Interesting applications include momentum maps for various automorphism groups which take values in Cheeger-Simons differential characters of the underlying manifolds.

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# MOMENTUM MAPS FOR AUTOMORPHISM GROUPS

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Joint work with Tobias Diez, MPI Leipzig

**Hamiltonian systems, from topology to applications through analysis I, October 8-12, 2018**

# ROUGH OVERVIEW OF THE RESULTS

• The space of smooth sections of a fiber bundle whose fibers are symplectic manifolds (called *symplectic fiber bundle*) carries a natural symplectic structure. The group of bundle automorphisms acts symplectically on this space but does not admit a classical momentum map.

• We introduce a new concept of a group-valued momentum map, inspired by the Poisson Lie setting. The group-valued momentum map assigns to every section of the symplectic fiber bundle a principal circle-bundle. We study the properties of this group-valued momentum map.

• Many examples can be handled with this new momentum map:

◇ obtain generalized Clebsch variables for fluids with integral helicity;

◇ the anti-canonical bundle is the momentum map for the action of symplectomorphisms on the space of compatible complex structures;

◇ the Teichmüller moduli space is realized as a symplectic orbit reduced space associated to a coadjoint orbit of $SL(2,\mathbb{R})$ and spaces related to the other coadjoint orbits are identified and studied;

◇ the momentum map for the group of bundle automorphisms on the space of connections over a Riemannian surface encodes, besides the curvature, also topological information of the bundle.

# DETAILED OVERVIEW OF THE RESULTS

Noether's theorem states that every symmetry of a system generates a conservation law. In symplectic geometry formulation, these conserved quantities are encoded in the momentum map.

The momentum map is not only important in dynamical systems but is also a valuable tool in the study of differential geometric questions. Atiyah and Bott [1983] showed that the curvature of a connection on a principal bundle over a Riemann surface furnishes the momentum map for the action of the group of gauge transformations. They applied Morse theory to the norm-squared of the momentum map (the Yang–Mills functional) in order to obtain the cohomology of the moduli space of Yang–Mills solutions which, by the Narasimhan–Seshadri theorem, can be identified with the moduli space of stable holomorphic structures.

Later, Fujiki [1992] and Donaldson [1997, 1999, 2003] provided a momentum map picture for the relationship between the existence of constant scalar curvature Kähler metrics and stability in the sense of geometric invariant theory.

*FIRST GOAL: Provide a framework which encompasses the gauge theory setting of Atiyah and Bott [1983] together with the action of diffeomorphism groups of Fujiki [1992] and Donaldson [1997].*

Starting point is a symplectic fiber bundle of the form $F = P \times_G \underline{F}$ for a principal $G$-bundle $P \to M$, where the typical fiber $\underline{F}$ is endowed with a $G$-invariant symplectic form. The fiberwise symplectic structure, combined with a volume form on the base $M$, induces a symplectic form $\Omega$ on the space $\mathscr{F}$ of sections of $F \to M$. The gauge group of $P$ acts in a natural way on $\mathscr{F}$, leaving the induced symplectic form $\Omega$ invariant.

We show that the action possesses a momentum map which is completely determined by the momentum map of the $G$-action on the fiber $\underline{F}$.

If the bundle $P$ is natural, i.e., it comes with a lifted Diff$(M)$-action to bundle automorphisms (for example, this is the case when $P$ is the frame bundle of $M$), then the group of volume-preserving diffeomorphisms acts on the space $\mathscr{F}$ of sections and leaves $\Omega$ invariant. A precise statement will be given later.

There are essentially two contributions to the momentum map. The first term is the pull-back of the fiberwise symplectic structure. The second term involves the fiber momentum map and, morally speaking, captures how much the lift of diffeomorphisms to bundle automorphisms shifts in the vertical direction. The interesting point is that the momentum map for the automorphism group on the *infinite-dimensional* space of sections is canonically constructed from the *finite-dimensional* symplectic $G$-manifold $\underline{F}$.

In contrast to the case of the action of the gauge group, the momentum map for the symplectic action of the diffeomorphism group on the space of sections *does not exist in full generality*. Donaldson [2000, 2003] already pointed this out. The obstruction has a topological character, i.e., certain cohomology groups have to vanish. To remedy this situation, one usually restricts the actions to certain "exact" subgroups, e.g., the subgroup of Hamiltonian diffeomorphisms in the group of all symplectomorphisms. Then, the actions of these subgroups do admit classical momentum maps.

Working from a completely different point of view, similar observations were made by Gay-Balmaz and Vizman [2012] in their study of the classical dual pair in hydrodynamics. In this case, the symplectic action of volume-preserving diffeomorphisms on a symplectic manifold of mappings only has a momentum map under certain topological conditions and one is forced to work with suitable central extensions of the group of exact volume-preserving diffeomorphisms.

OUR POINT OF VIEW: The above mentioned topological obstructions are not a bug but a feature of the theory. The action of the diffeomorphism group interacts with, and is largely determined by, the topological structure of the bundle. Thus, one would expect to capture certain topological data (like characteristic classes) that are "conserved" by the action and such "conservation laws" should be encoded in the momentum map. Since the classical momentum map takes values in a continuous vector space, there is no space to "store" discrete topological information. Hence, whenever those classes do not vanish, the momentum map does not exists.

*SECOND GOAL: Turn these philosophical remarks into explicit mathematical statements. In order to do this, we generalize the notion of momentum maps allowing them to take values in groups.*

Our concept of a group-valued momentum map is inspired by the Poisson Lie momentum map introduced in the 1990 thesis of Lu and in Lu and Weinstein [1990]. The group-valued momentum map introduced here is a vast generalization of many notions of momentum maps appearing in the literature including circle-valued, cylinder-valued, and Lie algebra-valued momentum maps. We show that our generalized group-valued momentum map always exists for the action of the diffeomorphism group, without any topological assumptions on the base but some integrability conditions on the fiber model. The resulting momentum map captures topological invariants of the geometry, exactly in (the dual of) those cohomology classes which prevented the existence of a classical momentum map. This approach of extending the definition of the momentum map, besides the situation described above in Poisson geometry, in order to capture conservation laws not available using the classical definition, has been used successfully before in the theory of the cylinder-valued and optimal momentum maps (Ortega and Ratiu [2003]).

**Hydrodynamic example:** Let $(M,\mu)$ be a closed (i.e., compact connected, boundaryless) $n$-manifold with volume form $\mu$ (identified with the measure it defines, also denoted by $\mu$) and $(F,\omega)$ a symplectic manifold. The space $C^\infty(M,F)$ of smooth maps from $M$ to $F$ carries the weak symplectic form

$$\Omega_\phi(X,Y)=\int_M \omega_{\phi(m)}(X(m),Y(m))\,d\mu(m),$$

where $\phi\in C^\infty(M,F)$ and $X,Y\in \mathsf{T}_\phi C^\infty(M,F)$, i.e., $X,Y:M\to \mathsf{T}F$ satisfy $X(m),Y(m)\in \mathsf{T}_{\phi(m)}F$ for all $m\in M$. The natural action by precomposition of $\mathsf{Diff}_\mu(M)$, the group of diffeomorphisms of $M$ preserving the volume form $\mu$, leaves $\Omega$ invariant. If $\omega$ is exact, say with primitive $\vartheta$, then the momentum map assigns the 1-form $\phi^*\vartheta$ to a map $\phi\in C^\infty(M,F)$. Here, the space of volume-preserving vector fields $\mathfrak{X}_\mu(M)$ (the vector fields whose $\mu$-divergence vanishes) is identified with closed $(n-1)$-forms, i.e., $\mathfrak{X}_\mu(M)^*=\Omega^1(M)/\mathrm{d}\Omega^0(M)$. More generally, Gay-Balmaz and Vizman [2012] showed that a (non-equivariant) momentum map also exists when the pull-back of $\omega$ by all maps $\phi\in C^\infty(M,F)$ is exact; for example, this happens when $\mathsf{H}^2(M)$ is trivial.

Our generalized group-valued momentum map takes no longer values in $\mathfrak{X}_\mu(M)^*$, but instead in the Abelian group $\widehat{\mathsf{H}}^2(M, \mathsf{U}(1))$ that parametrizes principal circle bundles with connections modulo gauge equivalence. If $(F, \omega)$ has a prequantum bundle $(L, \vartheta)$, then our group-valued momentum map sends $\phi$ to the pull-back bundle with connection $\phi^*(L, \vartheta)$. We see that no (topological) restrictions have to be made for $M$ and only the integrability condition of the symplectic form $\omega$ is needed for the existence of a group-valued momentum map. In contrast to the classical momentum map, a $\widehat{\mathsf{H}}^2(M, \mathsf{U}(1))$-valued momentum map contains topological information. First, the Chern class of the bundle, as a class in $\mathsf{H}^2(M, \mathbb{Z})$, is available from the generalized momentum map. In our simple example, this is just the integral refinement of $\phi^*\omega$. A second class in $\mathsf{H}^1(M, \mathsf{U}(1))$ is related to the equivariance of the momentum map; we will make all of this precise later on.

| | Space | Action of | Chern class | Secondary topological class |
|---|---|---|---|---|
| Hydrodynamics | $\mathrm{C}^\infty(M, F)$ | $\mathrm{Diff}_\mu(M)$ | 0 (total vorticity) in $\mathrm{H}^2(M, \mathbb{Z})$ | Circulations in $\mathrm{H}^1(M)$ |
| Lagrangian embeddings | $\mathrm{C}^\infty(L, M)$ | $\mathrm{Diff}_\mu(L)$ | Torsion class in $\mathrm{H}^2(M, \mathbb{Z})$ | Liouville class in $\mathrm{H}^1(M)$ |
| Kähler geometry | $\Gamma^\infty(\mathrm{L}M \times_{\mathrm{Sp}} \mathrm{Sp}/\mathrm{U})$ | $\mathrm{Diff}_\omega(M)$ | $c_1(M) \cup [\omega]^{n-1}$ in $\mathrm{H}^{2n}(M, \mathbb{Z})$ | |
| Quantomorphism | $\Gamma^\infty(P \times_{\mathrm{U}(1)} \mathbb{C})$ | $\mathrm{Aut}_\Gamma(P)$ | trivial | |
| Gauge theory | $\mathrm{Conn}(P)$ | $\mathrm{Aut}(P)$ | Torsion class in $\mathrm{H}^{2n}(M, \mathbb{Z})$ | |

Overview of the examples. Here, $\mu$ is a volume form and $\omega$ a symplectic form. Moreover, $Q \to M$ denotes a prequantum circle bundle with connection $\Gamma$ and $P \to M$ is an arbitrary principal $G$-bundle. The frame bundle is denoted by L$M$. We also abbreviated the homogeneous space Sp$(2n, \mathbb{R})$/U$(n)$ by Sp/U.

## Comments on the examples in the table

1.) Marsden and Weinstein [1983] construct Clebsch variables for ideal fluids starting from a similar infinite-dimensional symplectic system as discussed above. It turns out, that every vector field represented in those Clebsch variables has vanishing helicity, i.e., such a fluid configuration has trivial topology and no links or knots. Our more general framework allows to construct generalized Clebsch variables for vector fields with integral helicity.

2.) When applied to the space of Lagrangian immersions, the group-valued momentum map recovers the Liouville class as the topological data. Thus, we realize moduli spaces of Lagrangian immersions (and modifications thereof) as symplectic quotients.

3.) and 4.) Many examples with geometric significance are obtained when the typical fiber $\underline{F}$ is a symplectic homogeneous space $G/H$. In this case, sections of $\mathsf{L}M \times_G \underline{F}$ correspond to a reduction of the $G$-frame bundle $\mathsf{L}M$ to $H$.

Important case: the space of almost complex structures compatible with a given symplectic structure, i.e., $\underline{F} = \mathsf{Sp}(2n, \mathbb{R})/\mathsf{U}(n)$. In this case, the group-valued momentum map for the group of symplectomorphisms assigns to an almost complex structure the anti-canonical bundle. It was already observed by Fujiki [1992] and Donaldson [1997] that the Hermitian scalar curvature furnishes a classical momentum map for the action of the group of Hamiltonian symplectomorphisms. Of course, the Hermitian scalar curvature is the curvature of the anti-canonical bundle. Thus the group-valued momentum map combines the geometric curvature structure with the topological data of the anti-canonical bundle. For the case of a 2-dimensional base manifold, we realize the Teichmüller moduli space with the symplectic Weil–Petersson form as a symplectic orbit reduced space.

5.) We extend the classical setting of Atiyah and Bott [1983] in two ways. First, we generalize the gauge theoretic setting from 2-dimensional surfaces to arbitrary symplectic manifolds as the base (a similar extension was already discussed by Donaldson [1987]).

Secondly, and more importantly, we determine the group-valued momentum map for the action of the *full automorphism group* on the space of connections. Besides the curvature, the group-valued momentum map also encodes a torsion class in $\mathsf{H}^{2n}(M, \mathbb{Z})$, that arises from flat group homomorphisms $\mathsf{Sp}(2n, \mathbb{R}) \times G \to \mathsf{U}(1)$.

## General comments

1.) In contrast to most papers discussing infinite-dimensional symplectic geometry, we do not work formally, but really address the functional analytical problems arising from the transition to the infinite-dimensional setting. In particular, smoothness of maps between infinite dimensional manifolds is understood in the sense of locally convex spaces as, for example, presented in Neeb [2006] or the Russian school (Smolyanov).

2.) Integrality of certain symplectic forms play a central role. To a large extent, this assumption was made for convenience. Most results carry directly over to symplectic forms with discrete period groups per$\omega \subseteq \mathbb{R}$, without much technical effort. In spirit, our results also hold in the general setting without any assumptions on the period group; however, then one is forced to work in the diffeological category because the quotient $\mathbb{R}/\text{per}\omega$ may no longer be a Lie group.

3.) Most of our symplectic reduced spaces are obtained as (sometimes singular) *orbit* reduced spaces, a theory that is not yet present in the literature for infinite dimensional systems, even though we state theorems using it. However, the techniques in Diez [2018] which completely treats infinite dimensional singular symplectic *point* reduction, combined with the strategy in the book by Ortega and Ratiu [2003] for finite dimensional singular symplectic orbit reduction, yields a general theory of infinite dimensional singular symplectic orbit reduction, which is precisely what is needed here. This is the focus of a future paper.

# GROUP-VALUED MOMENTUM MAPS

Inspiration is the momentum map for Poisson Lie group actions. Our group-valued momentum map is *not* built on the pattern from the theory of quasi-Hamiltonian actions; in the Abelian case it extends this theory, but it is totally different for non-Abelian groups.

## Poisson Lie group momentum maps

All manifolds and Lie groups are finite-dimensional. The theory is due to Lu [1990], Lu and Weinstein [1990].

A Lie group $G$ is a *Poisson Lie group* if it is simultaneously a Poisson manifold and group multiplication and inversion are Poisson maps. Let $\varpi_G \in \mathfrak{X}^2(G)(=$ bivector fields) denote the Poisson tensor of $G$. Let $(M, \varpi_M)$ be a Poisson manifold.

A *Poisson action* of the Poisson Lie group $G$ on $(M, \varpi_M)$ is a smooth (left) action $G \times M \to M$ which is, in addition, a Poisson map (with $G \times M$ endowed with the product Poisson structure $\varpi_G \times \varpi_M$. (Functions on $G$ Poisson commute with functions on $M$).

The Poisson tensor $\varpi_G$ of a Poisson Lie group $G$, with Lie algebra $\mathfrak{g}$, necessarily vanishes at the identity element $e \in G$, which then allows for the definition of the *intrinsic derivative* $\epsilon : \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$ by $\epsilon(A) := (\mathfrak{L}_X \varpi_G)_e$, where $X \in \mathfrak{X}(G)$ is an arbitrary vector field satisfying $X_e = A$ and $\mathfrak{L}_X$ denotes the Lie derivative in the direction $X$. The dual map $\epsilon^* : \mathfrak{g}^* \wedge \mathfrak{g}^* \to \mathfrak{g}^*$ satisfies the Jacobi identity, thus endowing $\mathfrak{g}^*$ with a Lie algebra structure. The unique connected and simply connected Lie group $G^*$ whose Lie algebra is $\mathfrak{g}^*$ is called the *dual group* of $G$. The Lie group $G^*$ has a unique Poisson structure $\varpi_{G^*}$ relative to which $G^*$ is a Poisson Lie group such that the intrinsic derivative of $\varpi_{G^*}$ is the Lie bracket on $\mathfrak{g}$. If $G$ is connected and simply connected, the intrinsic derivative $\epsilon$ is a cocycle which uniquely determines both Poisson Lie tensors $\varpi_G$ and $\varpi_{G^*}$.

Let $G \times M \to M$ be a left Poisson action of the Poisson Lie group $(G, \varpi_G)$ on the Poisson manifold $(M, \varpi_M)$. A smooth map $J : M \to G^*$, if it exists, is called a *momentum map* of this action if

$$A^* + \varpi_M\left(\cdot, J^* A^l\right) = 0, \quad \text{for all} \quad A \in \mathfrak{g}.$$

Here, $A^*$ denotes the fundamental (or infinitesimal generator) vector field on $M$ induced by the infinitesimal action of $A \in \mathfrak{g}$, i.e.,

$$A^*(m) := \left.\frac{d}{dt}\right|_{t=0} \exp(tA) \cdot m, \quad \text{for all} \quad m \in M,$$

where $g \cdot m$ denotes the action of $g \in G$ on $m \in M$. The second term in the definition is interpreted in the following way. Since $\mathfrak{g}$ is the dual of $\mathfrak{g}^*$ (which is the Lie algebra of $G^*$), we think of $A$ as a linear map on $\mathfrak{g}^*$ and hence it defines a unique left invariant one-form $A^l \in \Omega^1(G^*)$ whose value at the identity is $A$, i.e., $(A^l)_a(v) = \left\langle A, \mathsf{L}_{a^{-1}} v \right\rangle$ for every $a \in G^*$ and $v \in \mathsf{T}_a G^*$, where $\mathsf{L}_{a^{-1}}$ denotes both the left translation by $a^{-1} \in G^*$ in $G^*$ and its tangent map on $\mathsf{T}G^*$.

Assume now that the Poisson manifold $(M, \varpi_M)$ is symplectic with symplectic form $\omega$ and let us unwind the definition in this case. For any $X_m \in \mathsf{T}_m M$ we have

$$\omega_m(A^*_m, X_m) = (\varpi_M)_m \left( \left(\varpi^\sharp_M\right)^{-1} A^*_m, \left(\varpi^\sharp_M\right)^{-1} X_m \right) = -\left(J^* A^l\right)_m X_m$$
$$= -\left(A^l\right)_{J(m)} (\mathsf{T}_m J(X_m)) = -\left\langle A, \mathsf{L}_{J(m)^{-1}} \mathsf{T}_m J(X_m) \right\rangle$$
$$= -\langle A, (\delta J)_m(X_m) \rangle ,$$

where $\mathsf{T}_m J : \mathsf{T}_m M \to \mathsf{T}_{J(m)} G^*$ is the derivative (tangent map) of $J : M \to G^*$ and $\delta J \in \Omega^1(M, \mathfrak{g}^*)$, defined by the last equality, is its *left logarithmic derivative*.

**Key observation:** The identity $\omega_m(A^*_m, X_m) + \langle A, (\delta J)_m(X_m) \rangle = 0$ proved above does not use the Poisson Lie group structure on $G$. This identity makes sense if the momentum map is replaced by a smooth map $J : M \to H$ with values in an arbitrary Lie group $H$, as long as there is a duality between the Lie algebras of $G$ and $H$. This observation leads to our generalization of Lu's momentum map. To define this generalization, we need a few preliminary concepts, inspired by their counterparts in the theory of Poisson Lie groups.

# Dual pairs of Lie algebras

A *dual pair of Lie algebras* (not necessarily finite dimensional) consists of two Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, which are in duality through a given (weakly) non-degenerate bilinear map $\kappa : \mathfrak{g} \times \mathfrak{h} \to \mathbb{R}$. Like in functional analysis, we often write the dual pair as $\kappa(\mathfrak{g}, \mathfrak{h})$. Intuitively, we think of $\mathfrak{h}$ as the dual vector space of $\mathfrak{g}$, endowed with its own Lie bracket operation, so sometimes we write $\mathfrak{g}^* := \mathfrak{h}$, even though $\mathfrak{g}^*$ is not necessarily the functional analytic dual of $\mathfrak{g}$.

Two Lie groups $G$ and $H$ are said to be *dual* to each other if there exists a bilinear form $\kappa : \mathfrak{g} \times \mathfrak{h} \to \mathbb{R}$ relative to which the associated Lie algebras are in duality. We use the notation $\kappa(G, H)$ in this case. As for Lie algebras, we often write $G^* := H$, intuitively thinking of $G^*$ as the dual group, as in the theory of Poisson Lie groups.

The notion of a dual pair of Lie algebras involves only the underlying vector spaces, while the Lie brackets play no role. We introduce a more rigid concept of duality, which takes all structures into account. For a given dual pair $\kappa(\mathfrak{g}, \mathfrak{h})$ of Lie algebras, define a bilinear skew-symmetric bracket on the double $\mathfrak{d} = \mathfrak{g} \times \mathfrak{h}$ by

$$[(A, \mu), (B, \nu)] = ([A, B]_{\mathfrak{g}} - \mathrm{ad}^*_{\mu} B + \mathrm{ad}^*_{\nu} A, [\mu, \nu]_{\mathfrak{h}} - \mathrm{ad}^*_{A} \nu + \mathrm{ad}^*_{B} \mu),$$

for $A, B \in \mathfrak{g}$, $\mu, \nu \in \mathfrak{h}$, where the infinitesimal coadjoint actions are defined with respect to $\kappa$ by

$$\kappa(B, \mathrm{ad}^*_{A} \mu) = \kappa([A, B]_{\mathfrak{g}}, \mu),$$
$$\kappa(\mathrm{ad}^*_{\mu} A, \nu) = \kappa(A, [\mu, \nu]_{\mathfrak{h}}).$$

However, this bracket does not satisfy the Jacobi identity, in general. A dual pair $\kappa(\mathfrak{g}, \mathfrak{h})$ of Lie algebras is called a *Lie bialgebra*, if this bracket on $\mathfrak{d} = \mathfrak{g} \times \mathfrak{h}$ is a Lie bracket. In this case, we denote the double by $\mathfrak{g} \bowtie \mathfrak{h}$.

**Examples** 1.) $\kappa(\mathfrak{g},\mathfrak{h})$ dual pair with $\mathfrak{h}$ Abelian, so the coadjoint action $\mathrm{ad}^*_\mu : \mathfrak{g} \to \mathfrak{g}$ is trivial for every $\mu \in \mathfrak{h}$. Hence the bracket on the double $\mathfrak{d}$ simplifies to $[(A,\mu),(B,\nu)] = ([A,B]_{\mathfrak{g}}, -\mathrm{ad}^*_A \nu + \mathrm{ad}^*_B \mu)$, so $\mathfrak{d}$ is the semidirect product $\mathfrak{g} \ltimes_{\mathrm{ad}^*} \mathfrak{h}$ of Lie algebras, where $\mathfrak{g}$ acts on $\mathfrak{h}$ by the $\kappa$-coadjoint action. The Jacobi identity always holds and thus $\kappa(\mathfrak{g},\mathfrak{h})$ is a Lie bialgebra.

2.) (Group of volume-preserving diffeomorphisms) $M$ be a compact manifold with a volume form $\mu$. Then the group $G = \mathrm{Diff}_\mu(M)$ of volume-preserving diffeomorphisms is a Fréchet Lie group (already known to Hamilton [1982]). Its Lie algebra is $\mathfrak{g} = \mathfrak{X}_\mu(M) := \{X \in \mathfrak{X}(M) \mid \mathfrak{L}_X \mu = 0\}$. Hence, we also identify $\mathfrak{X}_\mu(M)$ with $\Omega^{\dim M - 1}_{\mathrm{cl}}(M)$ via $X \mapsto \mathrm{i}_X \mu$, where $\Omega^k_{\mathrm{cl}}(M)$ denotes the space of closed $k$-forms on $M$. Thus $\Omega^1(M)/\mathrm{d}\Omega^0(M)$ is the regular dual with respect to the weakly non-degenerate integration paring

$$\kappa(X,\alpha) := (-1)^{\dim M - 1} \int_M (\mathrm{i}_X \mu) \wedge \alpha = \int_M (\mathrm{i}_X \alpha)\, \mu.$$

We now observe that a 1-form $\alpha$ can be interpreted as a trivial principal circle bundle with curvature $\mathsf{d}\alpha$. From this point of view, $\Omega^1(M)/\mathsf{d}\Omega^0(M)$ parametrizes equivalence classes of connections on a trivial principal circle bundle. Thus, it is natural to think of it as the Lie algebra of the Abelian group $H := \widehat{\mathsf{H}}^2(M, \mathsf{U}(1))$ of all principal circle bundles with connections, modulo gauge equivalence. This heuristic argument can be made rigorous using the theory of Cheeger-Simons differential characters. So we get a dual pair $\kappa(\mathsf{Diff}_\mu(M), \widehat{\mathsf{H}}^2(M, \mathsf{U}(1)))$ of Lie groups. For later use, it is convenient to introduce the notation $\widehat{\mathfrak{h}}^2(M, \mathsf{U}(1))$ for the Lie algebra $\Omega^1(M)/\mathsf{d}\Omega^0(M)$ of $\widehat{\mathsf{H}}^2(M, \mathsf{U}(1))$.

3.) (Group of symplectomorphisms) $(M, \omega)$ compact symplectic manifold. The group $G = \mathsf{Diff}_\omega(M)$ of symplectomorphisms is a Fréchet Lie group (Kriegl and Michor [1997]) with Lie algebra

$$\mathfrak{g} = \mathfrak{X}_\omega(M) := \{X \in \mathfrak{X}(M) \mid \mathsf{d}\mathsf{i}_X\omega = 0\} \ni X \stackrel{\sim}{\longleftrightarrow} \mathsf{i}_X\omega \in \Omega^1_{\mathsf{cl}}(M).$$

Thus the regular dual with respect to the natural integration paring

$$\kappa(X,\alpha) := \frac{(-1)^{\dim M-1}}{\left(\frac{1}{2}\dim M - 1\right)!}\int_M (\mathrm{i}_X\omega)\wedge\alpha$$

is $\widehat{\mathfrak{h}}^{\dim M}(M,\mathrm{U}(1)) := \Omega^{\dim M-1}(M)/\mathrm{d}\Omega^{\dim M-2}(M)$. The prefactor in front of the integral turns out to be a convenient choice in later computations. As in the case of volume-preserving diffeomorphisms, the Abelian Lie algebra $\widehat{\mathfrak{h}}^{\dim M}(M,\mathrm{U}(1))$ is integrated by the group $\widehat{\mathrm{H}}^{\dim M}(M,\mathrm{U}(1))$ of Cheeger-Simons differential characters with degree $\dim M$. These can be thought of as equivalence classes of circle $n$-bundles with connections in the sense of higher differential geometry.

**Remark:** Both $\mathrm{Diff}_\mu(M)$, $\mathrm{Diff}_\omega(M)$ have an Abelian dual group. Hence they are special cases of the first example; thus both dual pairs $(\mathfrak{X}_\mu(M),\widehat{\mathfrak{h}}^2(M,\mathrm{U}(1)))$ and $(\mathfrak{X}_\omega(M),\widehat{\mathfrak{h}}^{\dim M}(M,\mathrm{U}(1)))$ are Lie bialgebras.

Ignoring the particularities of the infinite-dimensional setting for a moment, Drinfeld's theorem states that there are essentially unique connected and simply connected Poisson Lie groups, whose Lie algebras are $\mathfrak{X}_\mu(M)$ and $\mathfrak{X}_\omega(M)$, respectively. We do not know if the groups $\mathsf{Diff}_\mu(M)$ or $\mathsf{Diff}_\omega(M)$ carry a non-trivial Poisson Lie structure integrating the above Lie bialgebras (this would require to find a non-trivial integration of the adjoint action). Moreover, we are not aware of *any* Poisson Lie structure on these groups, such that the actions used later in the examples are Poisson maps.

4.) (Gauge group) $P \to M$ right principal $G$-bundle, $M$ compact connected boundaryless. The group $\mathsf{Gau}(P)$ of gauge transformation is identified with the space of sections of $P \times_G G := (P \times G)/G$ and thus is a Fréchet Lie group with Lie algebra $\mathfrak{gau}(P) = \Gamma^\infty(\mathsf{Ad}\, P)$, the space of sections of the adjoint bundle $\mathsf{Ad}\, P := (P \times \mathfrak{g})/G$. Denote the dual of the adjoint bundle by $\mathsf{Ad}^* P := (P \times \mathfrak{g}^*)/G$, the action of $G$ on $\mathfrak{g}^*$ being the left coadjoint action.

The natural pairing

$$\kappa(\phi, \alpha) = \int_M \langle \phi, \alpha \rangle\,, \qquad \phi \in \mathfrak{gau}(P), \qquad \alpha \in \Omega^{\dim M}(M, \mathrm{Ad}^* P),$$

identifies $\Omega^{\dim M}(M, \mathrm{Ad}^* P)$ as the regular dual to $\mathfrak{gau}(P)$. In particular, if $M$ is endowed with a volume form $\mu$, then $\mathfrak{gau}^*(P) = \Gamma^\infty(\mathrm{Ad}^* P)$ is the dual by integration against $\mu$:

$$\langle \cdot, \cdot \rangle_{\mathrm{Ad}} : \Gamma^\infty(\mathrm{Ad}\, P) \times \Gamma^\infty(\mathrm{Ad}^* P) \to \mathbb{R}, \qquad (\rho, \varrho) \mapsto \int_M \langle \rho, \varrho \rangle\, \mu.$$

Moreover, an $\mathrm{Ad}_G$-invariant non-degenerate bilinear form $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ identifies $\mathfrak{gau}^*(P)$ with $\mathfrak{gau}(P)$ so that $\mathfrak{gau}(P)$ is self-dual in this case.

# Group-valued momentum maps

Let $M$ be $G$-manifold endowed with a symplectic form $\omega$ (not necessarily $G$-invariant). A *group-valued momentum map* is a pair $(J, \kappa)$, where $\kappa(G, G^*)$ is a dual pair of Lie groups and $J : M \to G^*$ is a smooth map satisfying

$$\mathrm{i}_{A^*}\omega + \kappa(A, \delta J) = 0, \quad \forall A \in \mathfrak{g};$$

$A^*$ is the fundamental vector field on $M$ induced by $A \in \mathfrak{g}$, $\delta J \in \Omega^1(M, \mathfrak{g}^*)$ is the left logarithmic derivative of $J$, $\mathfrak{g}$ is the Lie algebra of $G$, and $\mathfrak{g}^*$ is the Lie algebra of $G^*$.

**Examples:** 1.) (Standard momentum map) View $G^* = \mathfrak{g}^*$ as an Abelian group, $\kappa : \mathfrak{g} \times \mathfrak{g}^* \to \mathbb{R}$ duality pairing. Thus a $\mathfrak{g}^*$-valued momentum map is $J : M \to \mathfrak{g}^*$ satisfying the usual relation

$$\mathrm{i}_{A^*}\omega + \mathrm{d}J_A = 0, \quad \forall A \in \mathfrak{g}.$$

Here $J_A = \kappa(A, J) : M \to \mathbb{R}$ and the Abelian character of $\mathfrak{g}^*$ implies $\mathrm{d}J_A = \kappa(A, \mathrm{T}J) = \kappa(A, \delta J)$.

2.) (Lie algebra valued momentum map) If $\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ is a continuous, weakly non-degenerate, $\mathrm{Ad}_G$-invariant symmetric bilinear form, one identifies formally the functional analytic dual $\mathfrak{g}^*$ with $\mathfrak{g}$. So, a Lie algebra-valued momentum map is a smooth map $J : M \to \mathfrak{g}$ such that, for all $A \in \mathfrak{g}$, the component functions $J_A : M \to \mathbb{R}$ defined by $J_A(m) = \kappa(J(m), A)$, $m \in M$, satisfy

$$\mathbf{i}_{A^*}\omega + \mathbf{d}J_A = 0.$$

It is clear that such a Lie algebra-valued momentum map can be regarded as a group-valued momentum map with respect to the dual pair $\kappa(G, \mathfrak{g})$ where $\mathfrak{g}$ is viewed as an Abelian Lie group.

3.) (Poisson momentum map) $(M, \omega)$ finite-dimensional symplectic. Suppose a Poisson Lie group $G$ acts on $M$ in a Poisson manner. $G^*$ is the dual group. $\kappa : \mathfrak{g} \times \mathfrak{g}^* \to \mathbb{R}$ duality pairing. Hence a $G^*$-valued momentum map is a smooth map $J : M \to G^*$ satisfying

$$\mathbf{i}_{A^*}\omega + \kappa(A, \delta J) = 0.$$

As we have explained above, this is just a reformulation of the usual Lu momentum map relation in the context of symplectic geometry.

In other words, our group-valued momentum map is the natural generalization of the Poisson momentum map if the Lie group $G$ is not necessarily a Poisson Lie group.

4.) (Circle-valued momentum map) $(M, \omega)$ symplectic manifold, symplectic $G = \mathsf{U}(1)$-action. We let $G^* = \mathsf{U}(1)$ and take $\kappa : \mathfrak{u}(1) \times \mathfrak{u}(1) \to \mathbb{R}$, $\kappa(x, y) = xy$ (usual multiplication of real numbers under the identification $\mathfrak{u}(1) = \mathbb{R}$), as the pairing between the Lie algebras of $G$ and $G^*$. Thus, a map $J : M \to \mathsf{U}(1)$ is a group-valued momentum map if and only if

$$\mathbf{i}_{1^*}\omega + \delta J = 0,$$

i.e., recover the usual definition of a circle-valued momentum map.

5.) (Symplectic torus) $(V, \omega)$ symplectic vector space. A lattice $\Lambda$ in $V$ is a discrete subgroup of the additive group $(V, +)$. Thus $\Lambda$ acts naturally on $V$ by translations. The symplectic structure is invariant under this action and hence descends to a symplectic form $\omega_T$ on the torus $T = V/\Lambda$.

Since the translation action of $V$ commutes with the lattice action, the additive group $(V, +)$ also acts symplectically on the torus. The action of $V$ on itself has the momentum map

$$J : V \to V^*, \qquad v \mapsto \omega(v, \cdot).$$

However, $J$ is not invariant under the lattice action and so does *not* descend to a momentum map for the induced action of $V$ on the torus $T$. Indeed, it is well-known that, for cohomological reasons, the symplectic action on the torus does not admit a standard momentum map. Rather, $J$ transforms as

$$J(v + \lambda) = J(v) + \omega(\lambda, \cdot), \quad \lambda \in \Lambda.$$

Thus if $\omega(\lambda_1, \lambda_2) \in \mathbb{Z}$ holds for all $\lambda_1, \lambda_2 \in \Lambda$, then $J$ is equivariant with respect to the dual lattice action

$$\Lambda^* = \{\alpha \in V^* \mid \alpha(\lambda) \in \mathbb{Z} \text{ for all } \lambda \in \Lambda\}.$$

In this case, $J$ induces a $V^*/\Lambda^*$-valued momentum map $J_T$ on the torus $T$. It is interesting to note that the integrality condition $\omega(\lambda_1, \lambda_2) \in \mathbb{Z}$ is equivalent to $\omega_T$ being prequantizable.

6.) (Cylinder-valued momentum map) In a finite-dimensional context, Condevaux-Dazord-Molino [1988] introduced a momentum map with values in the cylinder $C := \mathfrak{g}^*/H$, where $H$ is the holonomy group of a flat connection on some bundle constructed in terms of the symplectic form and the action (in our language, $\alpha$ defined in the next section plays the role of the connection form). If the holonomy group $H$ is discrete, then $C$ is a Lie group with Lie algebra $\mathfrak{g}^*$. Thus $C$ is a dual group. Under the identification of the Lie algebra $\mathfrak{c} = \mathfrak{g}^*$, the cylinder-valued momentum map satisfies the defining identity for a group valued momentum map (shown in Ortega-Ratiu [2003], Theorem 5.2.8), and hence is a group-valued momentum map. The group-valued momentum map for the symplectic torus discussed in the previous example is also the cylinder-valued momentum map (shown in Example 5.2.5 of Ortega-Ratiu [2003].

The case when the holonomy group $H \subseteq \mathfrak{g}^*$ has accumulation points is pathological both in the framework of cylinder- as well as group-valued momentum maps; more on this later.

Despite its general nature, a group-valued momentum map still captures conserved quantities of the dynamical system, i.e., it has the *Noether property*.

Let $(M, \omega)$ be symplectic $G$-manifold. Suppose that the action has a $G^*$-valued momentum map $J : M \to G^*$. Let $h \in C^\infty(M)$ for which the Hamiltonian vector field $X_h$ exists and has a local flow. (Recall that vector fields on Fréchet manifolds do not need to have flows. This is more or less equivalent to local in time solutions of the corresponding partial differential equation.) If $h$ is $G$-invariant, then $J$ is constant along the integral curves of $X_h$.

# Existence and uniqueness

Define a $\mathfrak{g}^*$-valued 1-form $\alpha \in \Omega^1(M, \mathfrak{g}^*)$ by $\mathrm{i}_{A^*}\omega + \kappa(A, \alpha) = 0$, i.e.,

$$\kappa(A, \alpha_m(X_m)) = \omega_m(X_m, A^*_m) \quad \text{for all } X_m \in \mathrm{T}_m M,\ A \in \mathfrak{g}.$$

In infinite dimensions, the dual pairing $\kappa$ is mostly weakly non-degenerate. In such cases, there might not exists $\alpha \in \Omega^1(M, \mathfrak{g}^*)$ satisfying this identity, although $\alpha$ is unique, if it exists. We will assume from now on that we have such an $\alpha$.

The definition of the group-valued momentum map implies that the $G$-action on $M$ admits a $G^*$-valued momentum map if and only if $\alpha = \delta J$ for some smooth function $J : M \to G^*$, i.e., $\alpha \in \Omega^1(M, \mathfrak{g}^*)$ is log-exact.

Below, $[\alpha \wedge \beta]$ means the wedge product of the $\mathfrak{g}^*$-valued forms $\alpha$ and $\beta$ on $M$ associated to the bracket operation on $\mathfrak{g}^*$ (as the Lie algebra of $G^*$).

(i) $\alpha \in \Omega^1(M, \mathfrak{g}^*)$ on $M$ defined above satisfies the Maurer-Cartan equation

$$\mathrm{d}\alpha + \frac{1}{2}[\alpha \wedge \alpha] = 0$$

if and only if $\mathfrak{L}_{A^*}\omega = \frac{1}{2}\kappa(A, [\alpha \wedge \alpha])$ holds for all $A \in \mathfrak{g}$. In this case, we say that the $G$-action on $(M, \omega)$ is *$G^*$-symplectic*. If $G^*$ is Abelian, then the notions of symplectic and $G^*$-symplectic group actions coincide.
(ii) If the $G$-action on $M$ admits a $G^*$-valued momentum map $J : M \to G^*$, then $\delta J \in \Omega^1(M, \mathfrak{g}^*)$ satisfies the Maurer-Cartan equation.

Strengthen this statement in terms of the period map. Need the notion of regular Lie group.

A Lie group $G$ modeled on a locally convex space is *regular* if for each curve $c \in C^\infty([0,1],\mathfrak{g})$, the initial value problem $\delta\eta(t) := \mathsf{L}_{\eta(t)^{-1}}\dot{\eta}(t) = c(t)$, $\eta(0) = e$, has a solution $\eta_c \in C^\infty([0,1],G)$ and the endpoint evaluation map $C^\infty([0,1],\mathfrak{g}) \ni c \mapsto \eta_c(1) \in G$ is smooth. $G$ is regular $\Rightarrow$ it has a smooth exponential function. All Banach (so, in particular, all finite dimensional) Lie groups are regular.

Fix a point $m_0 \in M$ and consider a piece-wise smooth loop $\gamma : I \to M$ based at $m_0$. Pulling back $\alpha \in \Omega^1(M,\mathfrak{g}^*)$ by $\gamma$ yields a $\mathfrak{g}^*$-valued 1-form $\gamma^*\alpha$ on the interval $I$. Denote by $\eta_\gamma \in C^\infty(I,G^*)$ the solution of the initial value problem

$$\delta\eta = \gamma^*\alpha, \qquad \eta(0) = e,$$

which exists if $G^*$ is regular. Evaluating $\eta_\gamma$ at the endpoint 1, we obtain the *period homomorphism*

$$\mathrm{per}_\alpha : \pi_1(M,m_0) \ni [\gamma] \mapsto \eta_\gamma(1) \in G^*,$$

where $[\gamma]$ is the homotopy class of the loop $\gamma$.

Let $(M, \omega)$ be a connected symplectic manifold and $\kappa(G, G^*)$ a dual pair of Lie groups. In infinite dimensions, we additionally assume that $G^*$ is a regular Lie group. Suppose that the the framed formula has a solution $\alpha \in \Omega^1(M, \mathfrak{g}^*)$. Let $G$ act on $M$ in a $G^*$-symplectic way (i.e., $\alpha$ satisfies the Maurer-Cartan equation $\mathrm{d}\alpha + \frac{1}{2}[\alpha \wedge \alpha] = 0 \iff \mathfrak{L}_{A^*}\omega = \frac{1}{2}\kappa(A, [\alpha \wedge \alpha]), \forall A \in \mathfrak{g}$). Then there exists a $G^*$-valued momentum map if and only if the period homomorphism $\mathrm{per}_\alpha : \pi_1(M, m_0) \to G^*$ is trivial. Moreover, the momentum map is unique up to translation by a constant element $h \in G^*$.

**Example of a symplectic Lie group action without group valued momentum map:** Let $M = \mathbb{T}^2 = (\mathbb{R}/\mathbb{Z})^4 \ni (\varphi_1, \varphi_2, \psi_1, \psi_2)$. Endow $M$ with the symplectic form

$$\omega = \mathrm{d}\varphi_1 \wedge \mathrm{d}\varphi_2 + \sqrt{2}\mathrm{d}\psi_1 \wedge \mathrm{d}\psi_2.$$

The circle action given by $\lambda \cdot (\varphi_1, \varphi_2, \psi_1, \psi_2) = (\varphi_1 - \lambda, \varphi_2, \psi_1 - \lambda, \psi_2)$ is clearly symplectic. The framed equation has the solution

$$\alpha = \mathrm{d}\varphi_2 + \sqrt{2}\mathrm{d}\psi_2.$$

As the generators of $\pi_1(M)$ we take the four natural loops $\gamma_i$, where, for $1 \le i \le 4$, the loop $\gamma_i : I \to M$ winds once around the $i$-th circle in $M = (\mathbb{R}/\mathbb{Z})^4$. The pull-back of $\alpha$ by $\gamma_1$ and $\gamma_3$ vanishes and we find

$$\gamma_2^*\alpha = \mathrm{d}t \quad \text{and} \quad \gamma_4^*\alpha = \sqrt{2}\mathrm{d}t,$$

where $t \in I = [0,1]$. Since there are only two connected one-dimensional Lie groups, the only possible choices for the dual group are $G^* = \mathbb{R}$ and $G^* = \mathbb{R}/\mathbb{Z}$. In both cases, the initial value problem

$$\delta\eta = \gamma^*\alpha, \qquad \eta(0) = e$$

has the unique solutions $\eta_{\gamma_2}(t) = t$ and $\eta_{\gamma_4}(t) = \sqrt{2}t$. Thus neither for $G^* = \mathbb{R}$ nor for $G^* = \mathbb{R}/\mathbb{Z}$ the period homomorphism is trivial and hence no group-valued momentum map exists for this action.

This example exhibits another phenomenon that is particular for group-valued momentum maps: the action of a subgroup may not posses a group-valued momentum map even if the bigger group has a group-valued momentum map. In fact, the action $(\lambda_1, \lambda_2) \cdot (\varphi_1, \varphi_2, \psi_1, \psi_2) = (\varphi_1 - \lambda_1, \varphi_2, \psi_1 - \lambda_2, \psi_2)$ by $G = S^1 \times S^1$ has a group-valued momentum map (which is the product of two copies of the one discussed in example 6, the symplectic torus) but the action of the diagonally embedded circle has no group-valued momentum map as we have just seen.

# Equivariance and Poisson property

$\kappa(G, G^*)$ dual pair of Lie groups. $G$ acts on $(M, \omega)$ with group-valued momentum map $J : M \to G^*$. In which sense is $J$ equivariant? We assume, for simplicity, that $(G^*, +)$ is Abelian, identity element $0 \in G^*$. In our infinite dimensional examples this is always the case.

A left action $\Upsilon : G \times G^* \to G^*$ is called a *coconjugation action* if it integrates the coadjoint action, that is, $\delta_\eta \Upsilon_g(\eta \cdot \mu) = \mathrm{Ad}^*_{g^{-1}} \mu$, $\forall g \in G$, $\eta \in G^*$, and $\mu \in \mathfrak{g}^*$, where

• $\mathrm{Ad}^*$ is defined with respect to the duality pairing $\kappa$ by

$$\kappa(A, \mathrm{Ad}^*_g \mu) = \kappa(\mathrm{Ad}_g A, \mu),$$

• $\eta \cdot \mu := \mathsf{T}_e \mathsf{L}_\eta(\mu) \in \mathsf{T}_\eta G^*$,

• $\delta_\eta \Upsilon_g(\eta \cdot \mu) \in \mathfrak{g}^*$ denotes the (left) logarithmic derivative at $\eta$ of the map $\Upsilon_g : G^* \to G^*$ in the direction $\eta \cdot \mu \in \mathsf{T}_\eta G^*$.

If, moreover, $\Upsilon_g(\zeta + \eta) = \Upsilon_g(\zeta) + \Upsilon_g(\eta)$ holds for all $\zeta, \eta \in G^*$, then we say that the coconjugation action is *standard*.

A coconjugation does not always exist. Even it is exists, it does not have to be unique; nonetheless, the class of coconjugation actions is rather rigid.

Let $\Upsilon : G \times G^* \to G^*$ be a standard coconjugation action.
1.) If $\tilde{\Upsilon} : G \times G^* \to G^*$ is another coconjugation action (not necessarily standard), then there exists a map $c : G \times \pi_0(G^*) \to G^*$ such that $\tilde{\Upsilon} = \Upsilon + c$.
2.) Conversely, a map $c : G \to G^*$ defines a coconjugation action $\tilde{\Upsilon} := \Upsilon + c$ if and only if $c$ is a 1-cocycle with respect to $\Upsilon$, i.e., it satisfies $c(gh) = c(g) + \Upsilon_g(c(h))$ for all $g, h \in G$.

**Example:** The coadjoint representation is a standard coconjugation action of $G$ on $G^* := \mathfrak{g}^*$. Since $\mathfrak{g}^*$ is connected, the previous proposition establishes a bijection between coconjugation actions on $\mathfrak{g}^*$ and 1-cocycles $c : G \to \mathfrak{g}^*$. The coconjugation action corresponding to a 1-cocycle $c$ is the affine action $(g, \mu) \mapsto \mathrm{Ad}^*_{g^{-1}} \mu + c(g)$, which plays an important role for classical non-equivariant momentum maps (the Souriau cocycle).

In the classical setting $G^* \equiv \mathfrak{g}^*$ The derivative of a 1-cocycle $c : G \to \mathfrak{g}^*$ yields a 2-cocycle on the Lie algebra which in turn uniquely defines an affine Poisson structure on $\mathfrak{g}^*$. We will show that this extends to the general case and every coconjugation action gives rise to a Poisson structure on $G^*$.

Poisson structures in infinite dimensions need to be treated with caution! In our case, we can use the group structure to circumvent most of the technical issues. In finite dimensions, there are many equivalent ways to look at a bivector field on $G^*$. In a left-trivialization $\mathsf{T}G^* \simeq G^* \times \mathfrak{g}^*$, a bivector field $\pi_{G^*}$ is a smooth map $G^* \to \bigwedge^2 \mathfrak{g}^*$. Using reflexivity $\mathfrak{g}^{**} = \mathfrak{g}$, this may, equivalently, be viewed as a map

$$\pi_{G^*} : G^* \times \mathfrak{g} \to \mathfrak{g}^*.$$

It is this latter form that we adopt as the definition of a bivector field on $G^*$ in the infinite-dimensional setting.

Recall that for a smooth map $f : M \to G$ the *left logarithmic derivative* is the Lie algebra-valued 1-form $\delta f$ on $M$ defined by $\delta_m f(X) = f(m)^{-1} \cdot \mathsf{T}_m f(X)$, where $X \in \mathsf{T}_m M$.

Every standard coconjugation action $\Upsilon : G \times G^* \to G^*$ defines a Poisson Lie structure $\pi_{G^*} : G^* \times \mathfrak{g} \to \mathfrak{g}^*$ on $G^*$ by

$$\pi_{G^*}(\eta, A) = \delta_{(e,\eta)}\Upsilon(A, 0) = -\eta \cdot \mathsf{T}_e \Upsilon_\eta(A).$$

Moreover, if $\tilde{\Upsilon}$ is another coconjugation (not necessarily standard) and the map $c : G \times \pi_0(G^*) \to G^*$ from from the previous proposition satisfies $c_{\zeta+\eta} = c_\zeta + c_\eta - c_0$ for all $\zeta, \eta \in G^*$, then the associated bivector field $\tilde{\pi}_{G^*}$ is an affine Poisson structure.

By construction, $\Upsilon$ and the Poisson tensor $\pi_{G^*}$ are connected by

$$\eta \cdot \pi_{G^*}(\eta, A) = \mathsf{T}_e \Upsilon_\eta(A) \in \mathsf{T}_\eta G^*, \qquad \forall \eta \in G^*, \ \forall A \in \mathfrak{g}.$$

In Poisson geometry, actions satisfying this relation with respect to a Poisson structure on a Lie group are called *dressing actions*.

**Example:** Dual pair of Lie groups $\kappa(\mathsf{Diff}_\mu(M), \widehat{\mathsf{H}}^2(M, \mathsf{U}(1)))$. The coadjoint action of a diffeomorphism is given by pull-back. Thus a natural coconjugation is given by

$$\mathsf{Diff}_\mu(M) \times \widehat{\mathsf{H}}^2(M, \mathsf{U}(1)) \to \widehat{\mathsf{H}}^2(M, \mathsf{U}(1)), \quad (\phi, h) \mapsto (\phi^{-1})^* h.$$

The induced Poisson Lie structure on $\widehat{\mathsf{H}}^2(M, \mathsf{U}(1))$ is defined by

$$\widehat{\mathsf{H}}^2(M, \mathsf{U}(1)) \times \mathfrak{X}_\mu(M) \to \Omega^1(M)/\mathsf{d}\Omega^0(M), \quad (h, X) \mapsto [\mathsf{i}_X \mathsf{curv}_h].$$

Under the integration pairing, this corresponds to

$$\widehat{\mathsf{H}}^2(M, \mathsf{U}(1)) \times \mathfrak{X}_\mu(M) \times \mathfrak{X}_\mu(M) \to \mathbb{R} \quad (h, X, Y) \mapsto \int_M \mathsf{curv}_h(X, Y)\, \mu.$$

For fixed $h \in \widehat{\mathsf{H}}^2(M, \mathsf{U}(1))$, this is precisely the Lichnerowicz cocycle on $\mathfrak{X}_\mu(M)$ defined by the 2-form $\mathsf{curv}_h$. In other words, the Lichnerowicz cocycle is derived from the pull-back action. $\diamondsuit$

In the following, we also need the concept of the *left derivative* for maps whose domain is a Lie group, i.e., $F : G \to N$. The *left derivative of $F$ at $g \in G$ in the direction $A \in \mathfrak{g}$* is defined by

$$\mathsf{T}^{\mathsf{L}}_g F(A) := \mathsf{T}_g F(g \cdot A); \quad \text{thus} \quad \mathsf{T}^{\mathsf{L}}_g F : \mathfrak{g} \to \mathsf{T}_{F(g)} N \quad \text{linear.}$$

So, if $f : G \to \mathbb{R}$, then $\mathsf{T}^{\mathsf{L}}_g f : \mathfrak{g} \to \mathbb{R}$ is linear, i.e., i.e., $\mathsf{T}^{\mathsf{L}}_g f \in \mathfrak{g}^*$ .

Given coconjugation action, we say that the group-valued momentum map is *equivariant* if it is $G$-equivariant as a map $J : M \to G^*$. For classical momentum maps, there is a well-known equivalence between being equivariant and being a Poisson map. This is true for our group-valued momentum map.

For a smooth function $f : G^* \to \mathbb{R}$, the left derivative $\mathsf{T}^{\mathsf{L}}_{\eta} f : \mathfrak{g}^* \to \mathbb{R}$ at $\eta \in G^*$ is an element of the double dual $\mathfrak{g}^{**}$.

Need to reformulate the Poisson property without using $\mathfrak{g}^{**} = \mathfrak{g}$.

In finite dimensions, $\mathsf{T}^{\mathsf{L}}_{\eta} f \in \mathfrak{g}$, so if $f, g \in C^{\infty}(G^*)$,

$$\{f, g\}_{G^*}(\eta) = \kappa\left(\pi_{G^*}(\eta, \mathsf{T}^{\mathsf{L}}_{\eta} f), \mathsf{T}^{\mathsf{L}}_{\eta} g\right).$$

For $J : M \to G^*$, we calculate for any $X \in \mathsf{T}_m M$,

$$\begin{aligned} \mathrm{d}(f \circ J)(m)(X) &= \mathrm{d}f(J(m))(J(m) \cdot \delta_m J(X)) = \kappa(\mathsf{T}^{\mathsf{L}}_{J(m)} f, \delta_m J(X)) \\ &= -\omega_m((\mathsf{T}^{\mathsf{L}}_{J(m)} f) \cdot m, X). \end{aligned}$$

by the definition of the group-valued momentum map.

Applying the definition of the momentum map again

$$\begin{aligned}\{f\circ J, g\circ J\}_M(m) &= \varpi_m(\mathsf{d}(f\circ J)(m), \mathsf{d}(g\circ J))(m)\\ &= \omega_m((\mathsf{T}^{\mathsf{L}}_{J(m)}f)\cdot m, (\mathsf{T}^{\mathsf{L}}_{J(m)}g)\cdot m).\end{aligned}$$

So $J$ is Poisson map, i.e., $\{f\circ J, g\circ J\}_M = \{f,g\}_{G^*}\circ J$, if and only if

$$\omega_m(A\cdot m, B\cdot m) = \kappa(\pi_{G^*}(J(m), A), B), \quad \forall A, B\in\mathfrak{g}, \quad \forall m\in M.$$

This equation no longer relies on reflexivity to make sense and so we adopt it as the definition for $J$ to be a Poisson map. The left-hand side defines the so called *non-equivariance* cocycle $\sigma_m(A,B) = \omega_m(A\cdot m, B\cdot m)$. Thus $J$ is a Poisson map if and only if the Lie algebra cocycles $\sigma_m$ and $\kappa(\pi_{G^*}(J(m),\cdot),\cdot)$ coincide.

If $J: M\to G^*$ is equivariant with respect to a given coconjugation action then it is Poisson relative to the induced Poisson tensor $\pi_{G^*}$.

**Proof:** $J: M\to G^*$ equivariant $\Rightarrow J(g\cdot m) = \Upsilon_g J(m)$ ($\Upsilon$ coconjugation) Thus, $\delta_m J(A\cdot m) = \pi_{G^*}(J(m), A)$ by the definition of $\pi_{G^*}$. On the other hand, $\kappa(B, \delta_m J(A\cdot m)) = \omega_m(A\cdot m, B\cdot m)$ and the claim follows. $\square$

# Momentum maps for group extensions

Short exact sequence of Lie groups

$$e \longrightarrow H \stackrel{\iota}{\longrightarrow} K \stackrel{\pi}{\longrightarrow} G \longrightarrow e.$$

Suppose $K$ acts symplectically on $(M, \omega)$. Seek an expression of the momentum map for the $K$-action in terms of the momentum maps for the groups $H$ and $G$, assuming they exist. Similar questions occur in the context of symplectic reduction by stages.

The induced short exact sequence of Lie algebras

$$0 \longrightarrow \mathfrak{h} \stackrel{\iota}{\longrightarrow} \mathfrak{k} \stackrel{\pi}{\longrightarrow} \mathfrak{g} \longrightarrow 0$$

always splits as vector spaces but not necessarily as Lie algebras. Fix a splitting $\sigma : \mathfrak{g} \to \mathfrak{k}$ in the category of locally convex vector spaces and write $\mathfrak{h} \oplus_\sigma \mathfrak{g} = \mathfrak{k}$ for the corresponding direct sum. Thus, every $A \in \mathfrak{k}$ can be uniquely written as the sum $A = \iota(A_H) + \sigma(A_G)$ with $A_H \in \mathfrak{h}$ and $A_G = \pi(A) \in \mathfrak{g}$. Assume that $\mathfrak{h}$ is self-dual with respect to a pairing $\langle \cdot, \cdot \rangle$ and $\kappa(G, G^*)$ is a dual pair of Lie groups.

In this setting, if the induced action of $H$ on $M$ has a standard momentum map $J_H : M \to \mathfrak{h}$ with respect to the pairing $\langle \mathfrak{h}, \mathfrak{h} \rangle$ and there exists a map $J_\sigma : M \to G^*$ satisfying

$$\mathrm{i}_{\sigma(A_G)^*}\omega + \kappa(A_G, \delta J_\sigma) = 0, \qquad A_G \in \mathfrak{g},$$

then $J_K = (J_H, J_\sigma) : M \to \mathfrak{h} \times G^*$ is a group-valued $K$-momentum map with respect to the pairing $(\mathfrak{h} \oplus_\sigma \mathfrak{g}, \mathfrak{h} \oplus \mathfrak{g}^*) = \langle \mathfrak{h}, \mathfrak{h} \rangle + \kappa(\mathfrak{g}, \mathfrak{g}^*)$.

The identity in the statement is, formally, the momentum map relation for $G$. However, we do not assume that $\sigma$ is a splitting on the level of Lie algebras. Hence $G$, or its Lie algebra $\mathfrak{g}$, does not act on $M$ via $\sigma$ and $J_\sigma : M \to G^*$ *is not a momentum map*. However, if $G$ happens to act on $M$ through a different splitting $\chi : G \to K$ which is a group section of $\pi$, then $J_\sigma$ is the momentum map up to some twisting by $J_H$.

In this setting, let $\chi : \mathfrak{g} \to \mathfrak{k}$ be a Lie algebra homomorphism splitting the exact Lie algebra sequence above. Hence there is an induced (infinitesimal) $\mathfrak{g}$-action on $M$. Define $\tau_{\sigma\chi} := \chi - \sigma : \mathfrak{g} \to \mathfrak{h}$. Its dual map with respect to the chosen pairings is denoted by $\tau^*_{\sigma\chi} : \mathfrak{h}^* \to \mathfrak{g}^*$. Assume that the dual group $G^*$ of $G$ is Abelian. Then

$$J_\chi : M \to G^*, \qquad m \mapsto J_\sigma(m) \cdot \exp(\tau^*_{\sigma\chi} J_H(m))$$

is a group-valued momentum map for the $G$-action on $M$. Moreover, $J_\chi$ does not depend on the splitting $\sigma$.

Determine the momentum map for the action of a subgroup.

$G$ acts symplectically on $(M, \omega)$ with group-valued momentum map $J : M \to G^*$. Let $\iota : H \to G$ be a Lie group homomorphism; hence $H$ acts through $G$ on $M$. Fix a dual group $H^*$ of $H$. Suppose that there is a Lie group homomorphism $\rho : G^* \to H^*$ whose associated Lie algebra homomorphism $\rho : \mathfrak{g}^* \to \mathfrak{h}^*$ is the dual of $\iota : \mathfrak{h} \to \mathfrak{g}$ with respect to $\kappa(\mathfrak{g}, \mathfrak{g}^*)$ and $\langle \mathfrak{h}, \mathfrak{h}^* \rangle$. Then $J^H := \rho \circ J : M \to H^*$ is a group-valued momentum map for the induced $H$-action.

These assumptions on $\rho$ are automatically satisfied in finite-dimensions if $G$ is connected and $\pi_1(G, e) = 1$. In infinite dimensions, however, the adjoint $\rho : \mathfrak{g}^* \to \mathfrak{h}^*$ of the linear map $\iota$ does not need to exist, and even if it exists, it does not necessarily integrate to a Lie group homomorphism (for this, we would need some regularity assumptions on the dual group $H^*$ (Neeb [2006], Theorem III.1.5).