MSRI
Mathematical Sciences Research Institute

17 Gauss Way Berkeley, CA 94720-5070 p: 510.642.0143 f: 510.642.8609 www.msri.org

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Name: Tony Feng **Email/Phone:** tonyfeng@stanford.edu

Speaker's Name: Carlos Simpson

Talk Title: Infinity categories and why they are useful, I

Date: 2 / 1 / 19 **Time:** 3 : 30 am / [✓ pm] (circle one)

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INFINITY CATEGORIES AND WHY THEY ARE USEFUL, I

CARLOS SIMPSON

1. Motivation

The basic motivation is that we start with a Grothendieck site X. We know it is useful to think about sheaves on X, i.e. a functor $X^{op} \to \text{Set}$ satisfying a descent property.

It turns out that it is also desirable to look at "sheaves of spaces", i.e. functors $X^{op} \to \text{Spaces}$ that satisfy a descent property. The theory of such objects was constructed by Jardine, using ideas of Joyal. To manipulate these objects, we need some kind of theory of ∞-categories.

Similar problems had arisen previously, for example people wanted to consider sheaves valued in the derived category. In such contexts much progress was made without ∞-categories, although the apparatus of ∞-categories is useful.

2. Localization of categories

What's the basic phenomenon? The slogan is: "localization (of categories) creates homotopies".

Start with a category $\mathcal{C}$, and choose a subset of maps W in $\mathcal{C}$. One wants to make a category $W^{-1}\mathcal{C}$ by "adjoining inverses" to the maps in W.

Example 2.1. Consider a category with objects X, Y, Z and $u, v \colon X \to Y$ and $f \colon Y \to Z$ such that $fu = fv$. If we "invert" f, then we force $u = v$.

Suppose there's another object W and $g \colon Y \to W$ such that $gu = gv$, then if we invert g this also forces $u = v$. But now we've asked $u = v$ in two different ways. This creates an "S^1 in the space of maps". This could arise in "real life" in the following way.

You might have a fibered category $\mathcal{F} \to \mathcal{C}$ such that $f^* \colon \mathcal{F}_Z \xrightarrow{\sim} \mathcal{F}_Y$ and $g^* \colon \mathcal{F}_W \xrightarrow{\sim} \mathcal{F}_Y$. This will induce 2 natural isomorphism between the functors

$$u^* \colon \mathcal{F}_Y \xrightarrow{\sim} \mathcal{F}_X$$

$$v^* \colon \mathcal{F}_Y \xrightarrow{\sim} \mathcal{F}_X$$

which we denote $\psi_f \colon u^* \xrightarrow{\sim} v^*$ and $\psi_g \colon u^* \xrightarrow{\sim} v^*$. Combining them gives an automorphism $\psi_g^{-1} \circ \psi_f \colon u^* \xrightarrow{\sim} u^*$. This is the "loop around the S^1".

Then $\mathcal{F}$, viewed as a functor from $\mathcal{C}$, factors through the localization $W^{-1}\mathcal{C}$. The loop discussed earlier is a non-trivial 2-morphism in $W^{-1}\mathcal{C}$.

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In the world of ∞-categories, there is a localization operation $\mathcal{C}, W \mapsto W^{-1}\mathcal{C}$. Even if $\mathcal{C}$ is an ordinary category, the localization $W^{-1}\mathcal{C}$ will be an ∞-categories with homotopy in arbitrary degrees.

How does this relate to the localization we might be familiar with (e.g. derived categories), in which we don't deal with this? There is a truncation from ∞-categories to 1-categories. The usual localization is the composition of the ∞-categorical localization with this truncation, so it discards the higher homotopical information.

Example 2.2. For 2-categories, this operation crushes the category of maps between objects to its set of isomorphism classes.

The localization $W^{-1}\mathcal{C}$ was constructed by Dwyer-Kan, in fact in two ways: a general way, and the "hammock way".

3. SIMPLICIAL CATEGORIES

First of all, what is it? It's a *simplicial category.* One can think of simplicial categories as a model for ∞-categories. This means that it has a "set" (or maybe "Grothendieck universe") of objects, and for all objects x, y there is a simplicial set $A_*(x, y)$ of maps between x and y. (Morally we are imagining a space of maps, but it's technically better to use simplicial sets as a model for spaces.) Furthermore, there is a composition

$$A_*(x, y) \times A_*(y, z) \to A_*(x, z).$$

Recall that

$$|A_*(x, y)| \times |A_*(y, z)| = |A_*(x, y) \times A_*(x, z)|.$$

The point of the degeneracies is to make this true! Hence we get a map of spaces

$$|A_*(x, y) \times A_*(y, z)| \to |A_*(x, z)|.$$

Example 3.1. Let W be the weak equivalences in sSets. Let $\mathcal{S} = W^{-1}$sSets. Then $\mathcal{S}_*(X, Y)$ is a simplicial set representing maps from X to Y, i.e.

$$|\mathcal{S}_*(X, Y)| \sim \mathrm{Map}(|X|, |Y|).$$