Decompositions of Varieties, birationality, and K-theory

Inna Zakharevich

Scissor congruence with varieties:

$\mathbb{P}^1 = \mathbb{A}^1 \amalg *$, so $\mathbb{P}^1$ and $\mathbb{A}^1$ are "piecewise isom", or scissor congruent.

To understand this relation: introduce the following

Def. The grothendieck ring of varieties is

$$K_0(\mathrm{Var}) := \text{free ab group gen. by varieties} \Big/ \left. \begin{array}{l} Y \xhookrightarrow{\text{closed}} X \\ [X] = [Y] + [X \setminus Y] \end{array} \right\}$$

mostly for any base field in this talk!

ring structure: $[X][Y] = [X \times Y]$

Borisov: $\exists X, Y$ s.t. $[X] = [Y]$ but X, Y are not ~~~~ scissors congruent.

So $K_0(\mathrm{Var})$ does not quite encode scissors congruence.

In fact, hard to work with: - has zero divisors (Poonen)

- $[\mathbb{A}^1]$ is a zero divisor (Borisov)

But K_0 is the universal invariant

So we'd like to understand it.

From the view of topology: want K_0 to $\underbrace{\text{be}}_{\text{extend to}}$ part of K_n's, htpy groups of something

$\mathrm{Var} \supset \ldots \supset \mathrm{Var}^{(n)} \supset \mathrm{Var}^{(n-1)} \supset \ldots$

varieties of dimension at most $n-1$

Can define $K_0(\mathrm{Var}^{(n)})$ in the same way as above

$K_0(\mathrm{Var}^{(n)}) \longrightarrow K_0(\mathrm{Var})$ <u>NOT</u> injective! see next page.

$$K_0(\mathrm{Var}^{(n)}) / K_0(\mathrm{Var}^{(n-1)}) \simeq \mathbb{Z}\{B_n\}$$

birational iso classes of dim n.

Failure of injectivity:

$$\begin{tikzcd} \dim n \quad X \arrow[dr, hook] & & Y \arrow[dl, hook'] \quad \dim n \\ & W \text{ much larger} & \end{tikzcd}$$

$$\begin{tikzcd} \alpha: W \arrow[r, dashed, "\simeq", "\text{birational}"'] & W \\ U \arrow[u, hook] & V \arrow[u, hook] \end{tikzcd}$$

s.t. $W \setminus U \simeq X$, $W \setminus V \simeq Y$

Q: is this the only thing that can mess up injectivity?
is there some context in which $K_0 Var^{(n)} \subset K_0 Var$ constitute a filtration?

We want to introduce spaces (or spectra) w/ htpy groups the rings involved.
A filtration of spaces need not induce one on conn. components!

Crash course on Alg. K-theory: R a ring.

$K_0(R) :=$ free ab group gen. by f.d. proj. modules/R $\Big/$ $0 \to A \to B \to C \to 0$, $[B] = [A] + [C]$.

* same value for all fields; need more.

$K_1(R) := GL(R)^{ab}$ – contains information about automorphisms, which, by analogy, has to do with issues that arise w/ $K_0(Var^n) \to K_0(Var)$.

We want a space where (1) connected components remember projective modules
(2) paths remember ways of taking things apart and putting them together.

$Mod_R^{f.g.}$: A is "a part" of B if $A \hookrightarrow B$, $A \twoheadleftarrow B$ or some composite of these.

Def^n: $QMod_R^{fgp}$ has as objects f.g. proj R-modules

mor $A \twoheadleftarrow \cdot \hookrightarrow B$

How to define composites? Need to commute $A \hookrightarrow C \twoheadleftarrow B$.

Key observation:
$$\begin{tikzcd} & A\times B \arrow[dl, two heads] \arrow[dr, hook] & \\ A \arrow[dr, hook] & & B \arrow[dl, two heads] \\ & C & \end{tikzcd}$$

Consider the nerve of this category: $BQ\,\mathrm{Mod}_R$ is the simplicial set with

$$\left.\begin{array}{c} \text{objects as 0-cells} \\ \text{mors in cat as 1-cells} \\ A \twoheadleftarrow \hookrightarrow B \twoheadleftarrow \hookrightarrow C \text{ as 2-cells} \\ \vdots \end{array}\right\} \quad K(R) = \Omega BQ\,\mathrm{Mod}_R$$

<u>Theorem.</u> $\pi_0 K(R) = K_0(R)$
$\pi_1 K(R) = K_1(R)$
$\vdots$

Quillen's work and much following literature supports this is the right approach.
We want to do something similar for varieties.

<u>Question:</u> why do we need to take $K(R) = \Omega BQ\,\mathrm{Mod}_R$ instead of say $BQ\,\mathrm{Mod}_R$?

<u>A:</u> for any module A we have
$$\begin{tikzcd} & 0 \arrow[dl, two heads] \arrow[dr, hook] & \\ 0 & & A \\ & A \arrow[ul, two heads] \arrow[ur, hook, "="'] & \end{tikzcd}$$

So $\pi_0 BQ\,\mathrm{Mod}_R$ is trivial.

<u>But</u>, we can identify A with the path $A \twoheadleftarrow A \hookrightarrow A$ and check that π_1 is correct.

Morally: the relation on modules is a 3-term relation, which is a 2-cell relation on a 1-cell. So we need to pull 1-cells down to 0-cells by taking Ω.

What we need to mimick this construction is
(1) two sets of morphisms expressing "less than"
(2) a way to commute them past eachother.

Toy example: finite sets. $K(\mathrm{Fin\,Set})$.

ob: finite sets

(1) mor 1: injections
mor 2: injections

(2) start w/
$$A \xhookrightarrow{(i)} B \xhookrightarrow{(ii)} C$$
then
$$\begin{tikzcd} & A\cup(C\setminus B) \arrow[dr, hook, "(i)"] & \\ A \arrow[ur, hook, "(ii)"] \arrow[dr, hook, "(i)"'] & & C \\ & B \arrow[ur, hook, "(ii)"'] & \end{tikzcd}$$

and the two type (i)'s and type (ii)'s in the square have the same complements.

$Q\,\mathrm{Fin\,Set}$: ob finite sets
mor $A \xhookrightarrow{(ii)} \cdot \xhookleftarrow{(i)} B$

$$K(\mathrm{Fin\,Set}) = \Omega BQ\,\mathrm{Fin\,Set}.$$

Varieties:
objects: varieties
mar: closed embeddings, open embeddings $\rightsquigarrow$
$$\begin{tikzcd} & \cdot \arrow[dr, hook', "\text{closed}"] & \\ X \arrow[ur, hook, "\text{open}"] & & Z \end{tikzcd}$$
are morphisms.

commutation:
$$\begin{tikzcd} & X\cup(Z\setminus Y) \arrow[dr, hook, "\text{closed}"] & \\ X \arrow[ur, hook, "\text{open}"] \arrow[dr, hook, "\text{closed}"'] & & Z \\ & Y \arrow[ur, hook, "\text{open}"'] & \end{tikzcd}$$

$K(\mathrm{Var}) := \Omega BQ\mathrm{Var}$; can check $\pi_0 K(\mathrm{Var}) \simeq K_0(\mathrm{Var})$.

In general: a category where you can apply this sort of construction (the Q construction) is a CGW-category. (Campbell-Zakharevich.)

Note: Quillen's dévissage relating K-theory of a category to that of a subcategory really only seems to work for exact categories (seemed to be the case.)
But somehow, with slight assumptions, Quillen's dévissage and localization

work for CGW-categories.
(You need to assume kernels + cokernels.)

In fact, Quiller's proofs work!

<u>Applications</u>.

(1) observe that <u>dimension</u> gives a filtration on $K(Var)$ and in fact on $K_1(Var)$.

So, if we can compute filtration quotients, we'll get a SS converging to htpy of $K(Var)$.

$$K(Var^{(n)}) / K(Var^{(n-1)}) \simeq ?$$ need <u>Quiller's</u> <u>localization</u>.

<u>Quiller's localization</u>. If $\mathcal{A} \subseteq \mathcal{B}$ are ab. cats and $\mathcal{A}$ is a Serre subcat (so that $\mathcal{B}/\mathcal{A}$ has a nat'l ab. cat structure), then

$$K(\mathcal{A}) \longrightarrow K(\mathcal{B}) \longrightarrow K(\mathcal{B}/\mathcal{A})$$ is a homotopy fiber sequence.

$$\mathcal{A} \subseteq \mathcal{B} = Var^{(n-1)} \subseteq Var^{(n)}$$ satisfies this

<u>In spectra</u>: can compute $$K(Var^{(n)}) / K(Var^{(n-1)}) \simeq \bigvee_{\alpha \in B_n} \Sigma^\infty (B\,\mathrm{Birat\,Aut}(\alpha)_+)$$

morally speaking: decomposes as a sum of classifying spaces of birational automorphisms

<u>Compute SS</u>:

$$\pi_1 \Sigma^\infty B\,\mathrm{Birat\,Aut}_+(\alpha) \simeq \mathbb{Z}/2 \oplus \mathrm{Birat\,Aut}(\alpha)^{ab}$$

Differentials: ignore $\mathbb{Z}/2$, take $[\phi] \in \mathrm{Birat\,Aut}[X]$ to $[X \setminus U] - [X \setminus V]$

$$\phi : \underset{\subseteq X \supseteq}{U \longrightarrow V}$$

This shows that this is "the only thing that can go wrong" (see page 2).
Differentials enforce these relations + no others.

Note: since scissor relations aren't the only ones, we know there are nonzero differentials (but don't know where).

Theorem (Larsen-Lunts). $K_0(Var)/[\mathbb{A}] = \mathbb{Z}\left[\begin{smallmatrix}\text{stable birat.}\\ \text{isom. classes}\end{smallmatrix}\right]$ when $k=\mathbb{C}$.

Theorem (Zakharevich). Given $[X]-[Y] \in$ annihilator of $[\mathbb{A}^1]$,

$X \times \mathbb{A}^1$ and $Y \times \mathbb{A}^1$ are not piecewise isomorphic

(assuming they are chosen to have minimal dimension.)

Question: does this work for something other than $\mathbb{A}^1$?

A: for current proof, use birational factorization theorem. This factors a map as a series of blow-ups/blow-downs and relies on $\mathbb{A}^1$. So probably no.

this also constrains the base field. don't need $k=\mathbb{C}$, but do need this result or what Inna calls a "convenient field".