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Scissor congruence with varieties:

$\mathbb{P}^1 = \mathbb{A}^1 \amalg *$ , so  $\mathbb{P}^1$  and  $\mathbb{A}^1$  are "piecewise isom", or scissor congruent.

To understand this relation: introduce the following

Def. The Grothendieck ring of varieties is

$K_0(\text{Var}) :=$  free ab group gen. by varieties

$Y \xrightarrow{\text{closed}} X$

$[X] = [Y] + [X \setminus Y]$

mostly for any base field in this talk!

ring structure:  $[X][Y] = [X \times Y]$

Borisov:  $\exists X, Y$  s.t.  $[X] = [Y]$  but  $X, Y$  are not ~~not~~ scissors congruent.

So  $K_0(\text{Var})$  does not quite encode scissors congruence.

In fact, hard to work with: - has zero divisors (Ponzo)

-  $[\mathbb{A}^1]$  is a zero divisor (Borisov)

But  $K_0$  is the universal invariant

So we'd like to understand it.

From the view of topology: want  $K_0$  to be part of  $K_n$ 's, lumpy groups of something extended to

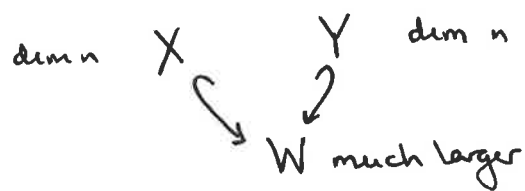
$\text{Var} \supset \dots \supset \text{Var}^{(n)} \supset \text{Var}^{(n-1)} \supset \dots$   
 $\uparrow$  varieties of dimension at most  $n-1$

Can define  $K_0(\text{Var}^{(n)})$  in the same way as above

$K_0(\text{Var}^{(n)}) \longrightarrow K_0(\text{Var})$  NOT injective! see next page.

$K_0(\text{Var}^{(n)}) / K_0(\text{Var}^{(n-1)}) \cong \mathbb{Z}\{B_n\}$   
 $\uparrow$  birational iso classes of dim  $n$ .

Failure of injectivity:



$$\alpha: W \xrightarrow{\cong} W$$

birationally

$$\begin{array}{ccc} \uparrow & & \uparrow \\ U & & V \end{array}$$

Zakharovich ②

s.t.  $W/U \cong X, W/V \cong Y$

Q: is this the only thing that can mess up injectivity?

is there some context in which  $K_0\text{Var}^{(n)}$   $\hookrightarrow$   $\text{CKVar}$  constitute a filtration?

We want to introduce spaces (or spectra) w/ fibering groups the rings involved.  
A filtration of spaces need not induce one on conn. components!

Crash course on Alg. K-theory:  $R$  a ring.

$$K_0(R) := \text{free ab group gen. by f.d. proj. modules}/R$$

\* some value for all fields; need more.

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

$$[B] = [A] + [C]$$

$K_1(R) := GL(R)^{ab}$  - contains information about automorphisms, which, by analogy, has to do with issues that arise w/  $K_0(\text{Var}^n) \leftrightarrow K_0(\text{Var})$ .

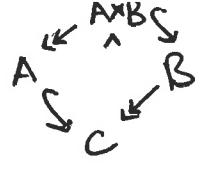
We want a space where (1) connected components remember projective modules  
(2) paths remember ways of taking things apart and putting them together.

f.g.  $\text{Mod}_R$ :  $A$  is "a part" of  $B$  if  $A \hookrightarrow B$  or some composite of these.  
 $A \leftarrow B$

Def<sup>n</sup>:  $\text{Mod}_R^{\text{f.g.}}$  has as objects f.g. proj  $R$ -modules

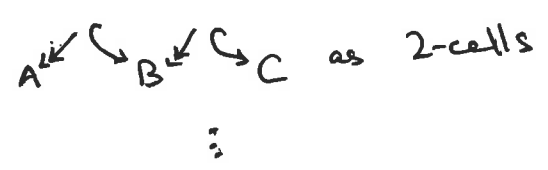
How to define composites? Need to commute  $A \leftarrow B$   $A \hookrightarrow C \leftarrow B$

Key observation:



Consider the nerve of this category:  $BQ Mod_R$  is the simplicial set with

objects as 0-cells  
morphisms as 1-cells



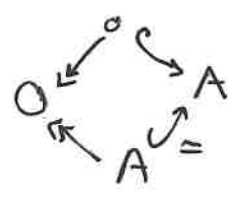
$$\left. \begin{array}{l} \text{objects as 0-cells} \\ \text{morphisms as 1-cells} \\ \text{A} \leftarrow \text{C} \leftarrow \text{B} \leftarrow \text{C} \text{ as 2-cells} \\ \vdots \end{array} \right\} K(R) = \int BQ Mod_R$$

Theorem.  $\pi_0 K(R) = K_0(R)$   
 $\pi_1 K(R) = K_1(R)$   
 $\vdots$

Quillen's work and much following literature supports this is the right approach. We want to do something similar for varieties.

Question: why do we need to take  $K(R) = \int BQ Mod_R$  instead of say  $BQ Mod_R$ ?

A: for any module A we have



So  $\pi_0 BQ Mod_R$  is trivial.

But, we can identify A with the path and check that  $\pi_1$  is correct.



Morally: the relation on modules is a 3-term relation, which is a 2-cell relation on a 1-cell. So we need to pull 1-cells down to 0-cells by taking  $\int$ .

What we need to mimick this construction is

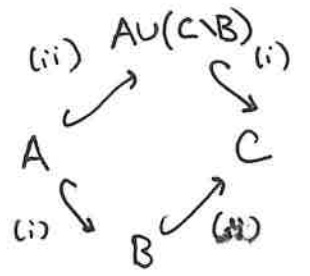
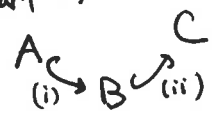
- (1) two sets of morphisms expressing "less than"
- (2) a way to compute them past each other.

Toy example: finite sets.  $K(\text{Fin Set})$ .

ob: finite sets

- (1) mor 1: injections
- mor 2: injections

(2) start w/ then



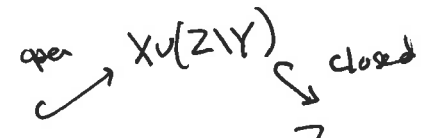
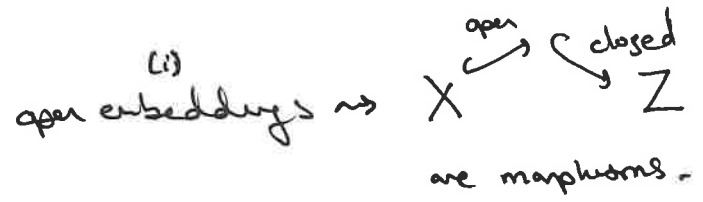
and the two type (i)'s and type (ii)'s in the square have the same complements.

$\mathcal{Q} \text{ Fin Set}$ : ob finite sets  
 mor  $A \xrightarrow{(ii)} C \xrightarrow{(i)} B$

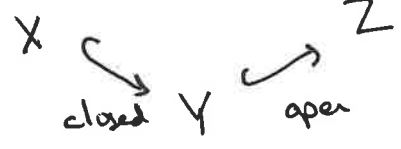
$K(\text{Fin Set}) = \mathcal{S} \mathcal{B} \mathcal{Q} \text{ Fin Set}$ .

Varieties:

objects: varieties  
 mor: closed embeddings, open embeddings



Commutativity:



$K(\text{Var}) := \mathcal{S} \mathcal{B} \mathcal{Q} \text{ Var}$ ; can check  $\pi_0 K(\text{Var}) \cong K_0(\text{Var})$ .

In general: a category where you can apply this sort of construction (the  $\mathcal{Q}$  construction) is a CGW-category. (Campbell-Zakharovich.)

Note: Quillen's dévissage relating K-theory of a category to that of a subcategory really only seems to work for exact categories (seemed to be the case.)  
 But somehow, with slight assumptions, Quillen's dévissage and localization

work for CGW-categories.

Zakharovich ⑤

(You need to assume kernels + cokernels.)

In fact, Quillen's proofs work!

### Applications.

(i) observe that dimension gives a filtration on  $K(\text{Var})$  and in fact on  $K_1(\text{Var})$ .

So, if we can compute filtration quotients, we'll get a SS converging to  $\text{htpy}$  of  $K(\text{Var})$ .

$$K(\text{Var}^{(n)}) / K(\text{Var}^{(n-1)}) \cong ? \quad \text{need Quillen's localization.}$$

Quillen's localization. If  $\mathcal{A} \subseteq \mathcal{B}$  are ab. cats and  $\mathcal{A}$  is a Serre subcat (so that  $\mathcal{B}/\mathcal{A}$  has a nat'l ab. cat structure), then

$$K(\mathcal{A}) \longrightarrow K(\mathcal{B}) \longrightarrow K(\mathcal{B}/\mathcal{A}) \text{ is a homotopy fiber sequence.}$$

$\mathcal{A} \subseteq \mathcal{B} = \text{Var}^{(n-1)} \subseteq \text{Var}^{(n)}$  satisfies this

In spectra: can compute  $K(\text{Var}^{(n)}) / K(\text{Var}^{(n-1)}) \cong \bigvee_{\alpha \in B_n} \Sigma^\infty (\text{BBirat Aut}(\alpha)_+)$

morally speaking: decomposes as a sum of classifying spaces of birational automorphisms

Compute SS:

$$\pi_1 \Sigma^\infty \text{BBirat Aut}_+(\alpha) \cong \mathbb{Z}/2 \oplus \text{Birat Aut}(\alpha)^{\text{ab}}$$

Differentials: ignore  $\mathbb{Z}/2$ , take  $[\phi] \in \text{Birat Aut}[X]$  to  $[X \cup U] - [X \cup V]$   
$$\phi: \begin{array}{ccc} U & \rightarrow & V \\ \cong & & \cong \\ X & \cong & X \end{array}$$

This shows that this is "the only thing that can go wrong" (see page 2).

Differentials enforce these relations + no others.

Note: since scissor relations aren't the only ones, we know there are nonzero differentials (but don't know where). Zakharovich (6)

Theorem (Larsen-Lunts).  $K_0(\text{Var})/[A] = \mathbb{Z} [\text{stable birat. isom. classes}]$  when  $k = \mathbb{C}$ .

Theorem (Zakharovich). Given  $[X] - [Y] \in \text{annihilator of } [A']$ ,  
 $X \times A'$  and  $Y \times A'$  are not piecewise isomorphic

(assuming they are chosen to have minimal dimension.)

Question: does this work for something other than  $A'$ ?

A: for correct proof, use birational factorization theorem. This factors a map as a series of blow-ups/blow-downs and relies on  $A'$ .  
So probably no.

→ this also constrains the base field. don't need  $k = \mathbb{C}$ , but do need this result or what Inna calls a "convenient field".