MSRI
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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Karol Kozul Email/Phone: kkozul@ualberta.ca

Speaker's Name: Stefan Patrikis

Talk Title: Drinfeld's work on the pro-semisimple completion of the fundamental group of a smooth variety over a finite field

Date: 4 / 12 / 19 Time: 3 : 30 am / pm (circle one)

Please summarize the lecture in 5 or fewer sentences:
The speaker presented Drinfelds thm about pro-semisimple completion, which is a type of "independence of ℓ" result. This can be used to define the notion of a compatible system in the geometric setting.

CHECK LIST

(This is **NOT** optional, we will **not** pay for **incomplete** forms)

- ☒ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☒ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - Computer Presentations: Obtain a copy of their presentation
  - Overhead: Obtain a copy or use the originals and scan them
  - Blackboard: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
  - Handouts: Obtain copies of and scan all handouts
- ☒ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☒ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- ☒ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

DRINFELD'S WORK ON PRO-SEMISIMPLE COMPLETION ...

— PATRIKIS

RECALL: • L. LAFFORGUE: $\forall$ CUSP. AUT REPS π OF (w/ FINITE ORDER CENTRAL CHAR)
$GL_n(\mathbb{A}_F)$ $(F = \mathbb{F}_q(X),\ X$ SM CURVE $/\mathbb{F}_q)$

$$\rightsquigarrow \{\sigma_{\pi,\lambda} : \Gamma_F \longrightarrow GL_n(\overline{\mathbb{Q}}_\lambda)\}_\lambda$$

COMPAT w/ π

↳ SPECIAL FEATURE OF GL_n: CAN APPLY CONVERSE THMS TO DEDUCE REVERSE DIRECTION

↳ IN PARTICULAR, GIVEN ANY IRR LISSE $\overline{\mathbb{Q}}_\lambda$ SHEAF $\mathcal{E}_\lambda$ ON X, $\exists$ λ'-ADIC COMPANIONS $\forall$ PLACE $\lambda' \nmid p$ OF $\overline{\mathbb{Q}}_{\lambda'}$

QUESTION: HOW TO CHARACTERIZE AND CONSTRUCT λ'-ADIC COMPANIONS OF A GIVEN $\sigma_\lambda : \pi_1(X) \longrightarrow \hat{G}(\overline{\mathbb{Q}}_\lambda)$?
(FOR GENERAL G)

LAST TALK: IF σ_λ HAS ZARISKI-DENSE IMAGE, THEN DEMANDING LOCAL COMPATIBILITY SUFFICES TO CHARACTERIZE λ'-ADIC COMPANIONS

DOWN-TO-EARTH INTERPRETATION OF DRINFELD'S THM YIELDS A WAY TO CHARACTERIZE + CONSTRUCT SUCH COMPANIONS IN GENERAL.

§ EXAMPLE OF THE PROBLEM

LET $H_n = \left\{ \begin{pmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{pmatrix} \subset GL_3(\mathbb{Z}/n) \right\}$

$$\begin{array}{ccccccccc} 1 \to & \mathbb{Z}/n & \to & H_n & \longrightarrow & (\mathbb{Z}/n)^2 & \longrightarrow & 1 \\ & \| & & A,B & \longmapsto & \text{GENS} & & \\ & \langle Z \rangle & & & & & & \\ & Z = [A,B] & & & & & & \end{array}$$

LET ζ = PRIM n^{TH} ROOT OF 1

FOR ANY $\alpha \in (\mathbb{Z}/n)^\times$, LET

$$\rho_\alpha : H_n \longrightarrow GL_n(\overline{\mathbb{Q}})$$

$$A \longmapsto \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 1 & & & 0 \end{pmatrix}$$

$$B \longmapsto \begin{pmatrix} 1 & & & \\ & \zeta^\alpha & & \\ & & \ddots & \\ & & & \zeta^{(n-1)\alpha} \end{pmatrix}$$

$$Z \longmapsto \zeta^\alpha \cdot id$$

LET $\overline{\rho}_\alpha$ BE PROJ'N, IE

$$\overline{\rho}_\alpha : H_n \longrightarrow PGL_n(\overline{\mathbb{Q}})$$

THEN FOR $\alpha \neq \beta$, $\overline{\rho}_\alpha$ AND $\overline{\rho}_\beta$ ARE LOCALLY, BUT NOT GLOBALLY, CONJUGATE

$$\begin{tikzcd} & & PGL_n(\overline{\mathbb{Q}}_\lambda) \\ & PGL_n(\overline{\mathbb{Q}}) \arrow[ur] & \\ H_n \arrow[ur, "\overline{\rho}_\alpha"] \arrow[dr, "\overline{\rho}_\beta"'] \arrow[uurr, bend left, "\rho_\lambda"] \arrow[ddrr, bend right, "\rho_{\lambda'}"'] & & \\ & PGL_n(\overline{\mathbb{Q}}) \arrow[dr] & \\ & & PGL_n(\overline{\mathbb{Q}}_{\lambda'}) \end{tikzcd}$$

$\rho_\lambda, \rho_{\lambda'}$ GIVE AN EXAMPLE OF THE ISSUE

FORMULATION OF MAIN THM

$X/\mathbb{F}_p$ SMOOTH CONNECTED VARIETY

LET $\Pi = \pi_1(X, \bar{x})$

DEF LET $\widehat{\Pi}_\ell$ BE THE ℓ-ADIC PRO-SEMISIMPLE COMPLETION OF Π, IE, $\varprojlim_{(G,\rho)} G$, WHERE $G/\mathbb{Q}_\ell$ IS A NOT NECESSARILY ~~SEMISI~~ CONNECTED SEMISIMPLE ALG GRP, AND $\rho : \Pi \longrightarrow G(\mathbb{Q}_\ell)$ W/ ZARISKI DENSE IMAGE

LIKEWISE, DEFINE FOR ANY PLACE $\lambda \nmid p$ OF $\overline{\mathbb{Q}}$, THE GP $\widehat{\Pi}_\lambda$ $(\cong \widehat{\Pi}_\ell \otimes \overline{\mathbb{Q}}_\lambda)$

DEF FOR ANY ALG CLOSED FIELD E LET $\text{PRO-SS}(E)$ = CAT OBJECTS ARE PRO-SS GRPS, MORPHISMS ARE

$$\text{HOM}_{\text{PRO-SS}(E)}(G_1, G_2) = \left\{ \begin{matrix} \text{GRP SCHEME ISOMS} \\ G_1 \xrightarrow{\sim} G_2 \end{matrix} \right\} \Big/ G_2^\circ\text{-CONJ}$$

LEMMA IF $E_1 \subset E_2$ ARE ALG CLOSED, THEN

$$\text{PRO-SS}(E_1) \xrightarrow[\text{BASE CHANGE}]{\sim} \text{PRO-SS}(E_2)$$

IN PART, $\hat{\Pi}_\lambda$ COMES FROM AN OBJECT

$$\hat{\Pi}_{(\lambda)} \in \text{PRO-SS}(\overline{\mathbb{Q}})$$

LET $\hat{\Pi}_\lambda \longrightarrow \Pi$ BE THE CANONICAL PROJ,

$\Pi \longrightarrow \hat{\Pi}_\lambda(\overline{\mathbb{Q}}_\lambda)$ THE CANONICAL REP

LET $\Pi_{Fr} = \bigcup_{x \in |X|} (Fr_x\text{-CONJ CLASS})^{\mathbb{N}}$

THEN WE HAVE

$$\begin{tikzcd} \Pi_{Fr} \arrow[r, hook] \arrow[drrr, dashed, "\text{THM}"'] & \Pi \arrow[r] & \hat{\Pi}_\lambda(\overline{\mathbb{Q}}_\lambda) \arrow[r] & {[\hat{\Pi}_{(\lambda)}](\overline{\mathbb{Q}}_\lambda)} \\ & & & {[\hat{\Pi}_{(\lambda)}](\overline{\mathbb{Q}})} \arrow[u, phantom, "\cup"] \end{tikzcd}$$

($[G] := G /\!/ G^\circ\text{-CONJ}$ FOR ANY PRO-REDUCTIVE G)

DIM $(X) = 1$: LAFFORGUE

DIM GENERAL : DRINFELD

MAIN THM X AS BEFORE. THEN $\forall \lambda, \lambda' \nmid p$, $\exists!$ ISOM IN PRO-SS$(\overline{\mathbb{Q}})$

$$\hat{\Pi}_{(\lambda)} \xrightarrow{\sim} \hat{\Pi}_{(\lambda')}$$

ST

$$\begin{tikzcd} & {[\hat{\Pi}_{(\lambda)}](\overline{\mathbb{Q}})} \arrow[dd, "\wr"] \arrow[dr] & \\ \Pi_{Fr} \arrow[ur] \arrow[dr] \arrow[r, phantom, "\circlearrowleft"] & {} \arrow[r, phantom, "\circlearrowleft"] & \Pi \\ & {[\hat{\Pi}_{(\lambda')}](\overline{\mathbb{Q}})} \arrow[ur] & \end{tikzcd}$$

THIS SUGGESTS A NOTION OF STRONGLY COMPATIBLE SYSTEM: FOR ANY ALG GP $H/\overline{\mathbb{Q}}$ EQUIPPED W/ A HOM

$$\Pi \xrightarrow{f} \pi_0(H),$$

CONSIDER

$$\mathrm{PAR}^{ss}(H,f) = \left\{ \begin{tikzcd} \underbrace{\hat{\Pi}}_{\substack{\text{ANY } \hat{\Pi}_{(\lambda)} \\ \in \text{PRO-SS}(\mathbb{Q})}} \arrow[r, "\rho"] \arrow[dr, "f"'] & H \arrow[d] \\ & \pi_0(H) \end{tikzcd} \right\} \Big/ H^\circ\text{-CONJ}$$

ASS'D TO SUCH A (H,f), $\rho \in \mathrm{PAR}^{ss}(H,f)$, WE GET

$$\{\rho_\lambda : \Pi \longrightarrow H(\overline{\mathbb{Q}}_\lambda)\}_\lambda$$

YIELDING A NOTION OF STRONGLY COMPATIBLE SYSTEM (AT LEAST FOR SEMI SIMPLE ZARISKI CLOSURES)

MAIN THM $\Longrightarrow$ ANY ρ_λ HAS STRONGLY COMPATIBLE COMPANIONS (W/ SS Z-CLOSURE)

MINOR VARIANT: ACCOUNTS FOR REDUCTIVE ZAR CLOSURES AS WELL (OR RED ZAR CLOSURE + PLAUSIBLY MOTIVIC)

STRATEGY OF PF OF MAIN THM

TOY QUESTION: IF G IS A CONNECTED RED'VE GRP:

THM (KAZHDAN – LARSEN – VARSHAVSKY) CAN RECOVER G FROM $K^+(G)$ = SEMIRING OF F.D. ALG REPS + ANY ISOM $K^+(G_1) \xrightarrow{\sim} K^+(G_2)$ COMES FROM ! $G_1 \xrightarrow{\sim} G_2$ UP TO CONJ

THM FAILS IF GRP IS DISCONNECTED :

ex $G = SL_{2n+1} \rtimes \underbrace{\mathbb{Z}/2}$

ACTING BY NONTRIV PINNED AUT (IE $g \mapsto g^{-T}$)

EVERY AUT OF G IS INNER $\rightsquigarrow$ INDUCE $id : K^+(G) \xrightarrow{\sim} K^+(G)$

BUT $\exists$ NON id ISOMS $K^+(G) \xrightarrow{\sim} K^+(G)$

COUNTEREX DOES NOT RESPECT λ-RING STR.

THIS MOTIVATES : PROVE RECONSTRUCTION THM FOR $\widehat{\Pi}_{(\lambda)}$ FROM THE λ-SEMIRINGS $K^+(\widehat{\Pi}_{(\lambda)} \times_{\Pi} U)$ FOR ANY OPEN SUBGP $U < \Pi$

FOR λ, λ', $K^+(\widehat{\Pi}_{(\lambda)} \times_{\Pi} U)$ IDENTIFY TO SAME
(λ')

λ-SUB-SEMIRING OF THE λ-SEMIRING OF FNS

$\Pi_{FR} \longrightarrow \overline{\mathbb{Q}}$

DRINFELD PROVES SUCH A THM