MODULI SPACES OF STABLE MAPS AND GW THEORY, II

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(*) Notes taken by Dhyan Aranha, all errors should be attributed to me and my ignorance about the subject. Corrections and suggestions are welcome, and should be sent to: dhyan.aranha@gmail.com.

We start where ended last time with the notion of cohomological field theory. Take, $(V, \eta, 1)$ (the state space), $\Omega = \{\Omega_{g,n}\}_{2g-2+n>0}$. We had a set of three axioms: (I) - Σ_n-invariance, (II)-splitting, (III)-forgetting the tail.

Where does the quantum cohomology come from? Whenever we have a cohomological field theory, say the one above, one gets a quantum product, $*$, on V. We call $(V, *, 1)$ the quantum cohomology ring and the way the product, $*$, is defined is:

$$\eta(v_1 * v_2, v_3) := \Omega_{0,3}(v_1, v_2, v_3) \in \mathbb{C}$$

(we can define the product in terms of the pairing because η is non-degenerate)

Exercise: Check $(V, *, 1)$ is a unital associative algebra. (Hint: Use $\overline{\mathcal{M}}_{0,4}$).

Givental-Teleman: We say that CohFT is semi-simple if $(V, *, 1)$ is semi-simple. Which means that $(V, *, 1)$ has a basis of idempotents.

In the world of CohFT's some of them come from GW-theory, but some don't. Similarly some are semi-simple and some are not, and also the collection of semi-simple ones doesn't necessarily coincide with the collections of ones coming from GW-Theory.

Let's fix a CohFT, Ω. with state space $(V, \eta, 1)$. Let

$$R(z) := \mathrm{id} + zR_1 + z^2 R_2 + \cdots$$

where

$$R_m \in End(V).$$

sometimes people write

$$R(z) \in \mathrm{id} + zEnd(V)[[z]].$$

Anyway, they should satisfy the so called symplectic condition:

$$R(z) \cdot R^*(-z) = \mathrm{id}$$

(where $R^*(z)$ means the adjoint with respect to η).

Definition 0.0.1. *The Givental group is then the collection of all such $R(z) \in \mathrm{id} + zEnd(V)[[z]]$ which satisfy the symplectic condition.*

Given an element R in the Givental group we can form a new CohFT, which we'll denote as $R\Omega$. So we have to say how to define $R\Omega_{g,n}(v_1,\dots,v_n)$. Roughly,

$$R\Omega_{g,n}(v_1,\dots,v_n) := \sum_{\Gamma\in G_{g,n}} \frac{1}{|Aut(\Gamma)|} i_{\gamma*}\Big(\prod_{vertices} C_v \prod_{legs} C_l \prod_{edges} C_e\Big)$$

where $G_{g,n} = \{\text{all dual graphs}\}$ and we have

(i) The C_v's are called the vertex contribution and is just

$$C_v := \Omega_{g(v),n(v)}$$

where $g(v)$ and $n(v)$ denote the genus and the number of half-edges and legs of the vertex.

(i) The C_l's are the leg contribution is the $End(V)$-valued cohomology class

$$C_l := R(\psi_l)$$

where $\psi_l \in H^2(\overline{\mathcal{M}}_{g(v),n(v)}, \mathbb{C})$ is the cotangent class at the marking corresponding to the leg.

(iii) The edge contribution is

$$C_e := \frac{\eta^{-1} - R(\psi_e')\eta^{-1}R(\psi_e'')^\top}{\psi_e' + \psi_e''}$$

where ψ_e' and ψ_e'' are the cotangent classes at the node which represents the edge e. The symplectic condition guarantees that this is well defined.

Remark 0.0.2. *For an in-depth explanation of the formula and definitions I (Dhyan) recommend "Cohomological field theory calculations" by Pandharipande; arxiv: https://arxiv.org/pdf/1712.02528.pdf.*

Result: The gadget $R\Omega$ is a CohFT, without the axiom (III) for the unit. (we'll fix this in a moment). Also, this defines a group action of Giv on those CohFT's which don't satisfy axiom (III).

There is another group action given via translation. Let

$$T = T_2 z^2 + T_3 Z^3 + \cdots \in V[[z]]$$

where $T_m \in V$. We define $T\Omega_{g,n}(v_1,\dots,v_n)$ as

$$T\Omega_{g,n}(v_1,\dots,v_n) := \sum_{m=0}^{\infty} \frac{1}{m!} p_{m*}(\Omega_{g,n+m}(v_1,\dots,v_n,T(\psi_{n+1}),\dots,T(\psi_{n+m}))$$

where $p_m : \overline{\mathcal{M}}_{g,n+m} \longrightarrow \overline{\mathcal{M}}_{g,n}$ forgets the last m markings.

Now we can define an action of the Givental-Teleman group on CohFT's: $R.\Omega = RT\Omega$ where $T(z) = z((\mathrm{id} - R(z)) * 1)$ the actions on the RHS are the ones we just defined.

Result: $R_\bullet\Omega$ is a CohFT with unit.

Statement of Givental-Teleman classification of semi-simple CohFT's : We only one more notion, which is called the topological part.

Let Ω be a CohFT with unit (i.e. satisfies axiom (III)). Then it is possible to define a new CohFT by taking the degree 0 part:

$$\omega_{g,n}(v_1, \ldots, v_n) = [\Omega_{g,n}(v_1, \ldots, v_n)]^0 \in H^0(\overline{\mathcal{M}}_{g,n}, \mathbb{C})$$

Note that $\omega_{g,n}$ is a cohomological field theory with unit (i.e. satisfies axiom (III)).

Theorem 0.0.3. *If Ω is a semi-simple* CohFT *with unit. Then there exists a unique $R \in Giv$ so that*

$$R_{\bullet}\omega = \Omega$$

Why is this useful? Well, because ω is much simpler than Ω. More precisely ω is determined just by $(V^0, *, 1)$ and η. Why is that?

$$\omega_{g,n}(v_1, \ldots, v_n) = [\Omega_{g,n}(v_1, \ldots, v_n)]^0 = \int_{\overline{\mathcal{M}}_{g,n}} \Omega_{g,n}(v_1, \ldots, v_n) \cdot [C, p_1, \ldots, p_n].$$

Then we can use the splitting axiom to compute. In the Gromov-Witten case this is the GW-theory with fixed complex structure on the domain.

<u>How to find R?</u>

<u>Example:</u> Solution of r-spin theory.

Let $r \geq 2$ integer, V a vector space of dimension $r-1$ with basis $\{e_0, \ldots, e_{r-2}\}$ and $\eta(e_a, e_b) = \delta_{a+b,r-2}$, $e_0 = 1$. This information determines the state space. Let $W^r_{g,n}$ denote the r-spin CohFT.

$$W^r_{g,n}(e_{a_1}, \ldots, e_{a_n}) \in H^*(\overline{\mathcal{M}}_{g,n})$$

which is usually called a Witten's class of degree

$$D^r_{g,n}(a_1, \ldots a_n) := \frac{(r-2)(g-1) + \sum a_i}{r}$$

(if it the numerator is not divisible by r then the class is zero). We have a moduli space of r-spin curves

$$\begin{array}{ccc} \overline{\mathcal{M}}^r_{g,(a_1,\ldots,a_n)} & & [C, p_1, \ldots, p_n, \mathcal{L}^{\otimes r} \simeq \omega_C(-\sum_{i=1}^r a_i p_i)] \\ \downarrow & & \\ \overline{\mathcal{M}}_{g,n} & & [C, p_1, \ldots p_n] \end{array}$$

Now take $g = 0$, we have (A. Pixton):

$$\# = \int_{\overline{\mathcal{M}}_{0,n}} W^r_n(A) = \frac{(n-1)!}{r^{n-3}} \dim[\rho_{r-2-a_1} \otimes \cdots \otimes \rho_{r-2-a_n}]^{\mathfrak{sl}_2}$$

where $A = (a_1, \ldots, a_n)$, $D^r_{0,n}(A) = n-3$ and the ρ_k is the k^{th} symmetric power of the standard 2-dimensional representation, ρ_1, of $\mathfrak{sl}_2$, that is $\rho_k = Sym^k(\rho_1)$.

It turns out as defined $W^r_{g,n}$ is not semi-simple. But there is a way to get around it. Let $\gamma \in V$ we can define a shifted r-spin theory

$$W^{r,\gamma}_{g,n}(v_1 \otimes \cdots \otimes v_n) = \sum \frac{1}{m!} p_{m*}(W^r_{g,n+m}(v_1, \ldots, v_n, \gamma, \ldots \gamma)$$

Now take $\gamma = (0, \ldots, 0, re_{r-2})$. Then we get a new CohFT: $W^{r,re_{r-2}}_{g,n}$.

Theorem 0.0.4. *(Pandharipande, Pixton, Zvonkine) The* CohFT, $W^{r,re_{r-2}}_{g,n}$ *is semi-simple.*

The way you prove it you calculate the algebra $(V, \hat{*}, 1) \simeq$ Verlinde algebra of level r for $\mathfrak{sl}_2$.
We can write down the idempotent basis for this algebra:

$$V_k = \sqrt{\frac{2}{r}} \sum_{a=0}^{r-2} \sin(\frac{(a+1)k\pi}{r}) e_a$$

and multiplcation

$$V_k \hat{*} V_l = \frac{\sqrt{\frac{r}{2}}}{\sin(\frac{k\pi}{r})} V_k \delta_{k,l}$$

where

$$\eta(V_k, V_l) = (-1)^{k-1} \delta_{k,l}.$$

Finally you can write down the topological part of the shifted thing:

$$\hat{\omega}^r_{g,n}(e_{a_1}, \ldots, e_{a_n}) = (\frac{r}{2})^{g-1} \sum_{k=1}^{r-1} \frac{(-1)^{(k-1)(g-1)} \prod_{i=1}^n \sin(\frac{(a_i+1)k\pi}{r})}{\sin(\frac{k\pi}{r})^{2g-2+n}}.$$

Now how to get the R matrix (This boils down to solving a differential equation)? We will give a characterization which is explicit in terms of hypergeometric series. If you do the case when $r = 3$ you get exactly 2 hyper geometric series that appear in Faber-Zagier, and Pixton's relations.

There is an Euler field:

$$\xi = \begin{pmatrix} & & & 2 \\ & & 2 & \\ & 2 & & \\ 2 & & & \end{pmatrix},$$

and then theres a grading operator:

$$\mu = \frac{1}{2r} \begin{pmatrix} -(r-2) & & & \\ & -(r-4) & & \\ & & \ddots & \\ & & & r-2 \end{pmatrix}.$$

The equation that determines the R matrix is

$$[R_{m+1}, \xi] = (m - \mu) R_m.$$

So to find the unique R matrix you have solve the last formula explicitly. You can almost never do this but in this example you can. Here is a hyper-geometric series

$$B_{r,a}(z) = \sum_{m=0}^{\infty} (\prod_1^m \frac{((2_i - 1)r - 2(a+1))((2_i - 1)r + 2(a+1))}{i})(\frac{-z}{16r^2})^m.$$

there is an even part, $B_{r,a}^{\text{even}}$ and an odd part $B_{r,a}^{\text{odd}}$ and

$$R_a^a(z) = B_{r,a}^{\text{even}}(z) \quad R_a^{r-2-a}(z) = B_{r,a}^{\text{odd}}(z).$$

We see for $r = 3$:

$$R(z) = \begin{pmatrix} B_{3,0}^{\text{even}} & B_{3,1}^{\text{odd}} \\ B_{3,0}^{\text{odd}} & B_{3,1}^{\text{even}} \end{pmatrix}$$

Conjecture: (Janda, Pandharipande, Pixton, Zvonkine)

$$r^{g-1} W_{g,n}^r(a_1, \dots, a_n) \in H^{2(g-1)}(\overline{\mathcal{M}}_{g,n})$$

such that $\sum a_i = 2g - 2$.

Theorem 0.0.5. *For all large r the expression above is a polynomial in r.*

Take constant term of $r^{g-1} W_{g,n}^r(a_1, \dots, a_n)$. This is a class in $H^{2(g-1)}(\overline{\mathcal{M}}_{g,n})$. The conjecture is that this class is $\overline{H}_{g,a_1,\dots,a_n} \subset \overline{\mathcal{M}}_{g,n}$.

This conjecture is going to be proved using results of Janda and Zvonkine.