In the field of Ehrhart theory, identification of lattice polytopes with unimodal Ehrhart h*-polynomials is a cornerstone investigation. The study of h*-unimodality is home to numerous long-standing conjectures within the field, and proofs thereof often reveal interesting algebra and combinatorics intrinsic to the associated lattice polytopes. Proof techniques for h*-unimodality are plentiful, and some are apparently more dependent on the lattice geometry of the polytope than others. In recent years, proving a polynomial has only real-roots has gained traction as a technique for verifying unimodality of h-polynomials in general. However, the geometric underpinnings of the real-rooted phenomena for h*-unimodality are not well-understood. As such, more examples of this property are always noteworthy. In this talk, we will discuss a family of lattice n-simplices that associate via their normalized volumes to the n^th-place values of positional numeral systems. The h*-polynomials for simplices associated to special systems such as the factoradics and the binary numerals recover ubiquitous h-polynomials, namely the Eulerian polynomials and binomial coefficients, respectively. Simplices associated to any base-r numeral system are also provably real-rooted. We will put the h*-real-rootedness of the simplices for numeral systems in context with that of their cousins, the s-lecture hall simplices, and discuss their admittance of this phenomena as it relates to other, more intrinsically geometric, reasons for h*-unimodality.

Ehrhart Unimodality and Simplices for Numeral Systems

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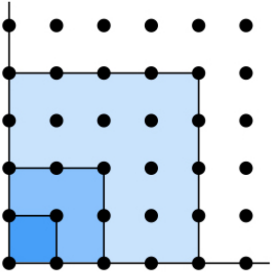
- $P \subset \mathbb{R}^n$ an d-dimensional lattice polytope.

- The Ehrhart series of P:

$$1 + \sum_{t \in \mathbb{Z}_{>1}} \mid tP \cap \mathbb{Z}^n \mid z^t = \frac{h_0^* + h_1^* z + \cdots + h_d^* z^d}{(1-z)^{d+1}}.$$



$P = [0,1]^2$

The Ehrhart h^*-polynomial of P:

$$h^*(P; z) := h_0^* + h_1^* z + \cdots + h_d^* z^d$$

- **Properties:**
  - $h^*(P; 1) = d! \operatorname{vol}(P)$
    $= $ normalized volume of P.
  - $h_1^* = \mid P \cap \mathbb{Z}^n \mid -d - 1.$
  - $h_0^*, \ldots, h_d^* \in \mathbb{Z}_{\geq 0}$ [Stanley, 1980]

- $\mid tP \cap \mathbb{Z}^2 \mid = (t+1)^2$
- $\sum_{t \in \mathbb{Z}_{\geq 0}} (t+1)^2 z^t = \frac{1+z}{(1-z)^3}$
- $h^*(P; z) = 1 + z$

$h^*(P;z)$ is "combinatorial."

If $a_0, a_1, \ldots, a_d \in \mathbb{Z}_{\geq 0}$ then maybe they coefficients of the polynomial

$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_d z^d$$

count a collection of combinatorial objects Ω as stratified by some parameter $k = 0, 1, \ldots, d$.

Question: When is $p(z)$ unimodal?

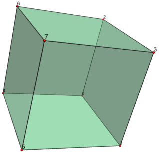
i.e., when is there a j such that $a_0 \leq \cdots \leq a_j \geq \cdots \geq a_d$?

- Unimodality is a distributional statement.
- Proofs can reveal hidden structure about Ω.
- variety of proof techniques exist [Stanley 1989, Brenti 1993, Brändén 2016]

- $h^*(P;z)$ is combinatorial.
- $h^*(P;z)$ arises via enumeration of lattice points in dilates of P.

Two natural questions:

- What different things does $h^*(P;z)$ count?
- When is $h^*(P;z)$ unimodal?



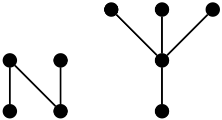
$P := [0,1]^n$

$$\sum_{t\geq 0}(t+1)^n z^t = \frac{A_n(z)}{(1-z)^{n+1}}$$

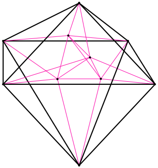
$$A_n(z) = \sum_{\pi\in S_n} z^{\mathrm{des}(\pi)},$$

n^{th} Eulerian Polynomial.

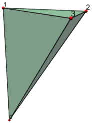
- **The Order Polytopes:** [Stanley, 1980]
  - Lipschitz Order Polytopes [Sanyal, Stump, 2015]
  - Double Poset Polytopes [Chappell, Friedel, Sanyal, 2016]
  - Twinned Order Polytopes [Hibi, Matsuda, Tsuchiya, 2015]

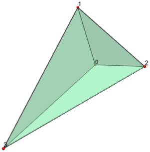


- **The (n, k)-hypersimplices:** [Katzman, 2005]
  - matroid polytopes [De Loera, Haws, Köppe, 2007]
  - r-stable hypersimplices [Braun, LS, 2014]
  - alcoved polytopes [Lam, Postnikov, 2007]

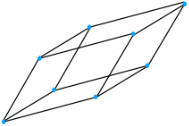


- **The s-lecture hall simplices:** [Savage, Schuster, 2012]
  - s-lecture hall order polytopes [Brändén, Leander, 2016]
  - simplices for numeral systems [LS, 2017]





- **Lattice Parallelpipeds:** [Schepers, Van Langenhoven, 2013]
  - Lattice Zonotopes [Beck, Jochemko, McCullough, 2016]



When is $h^*(P; z)$ unimodal?

How to answer this question:

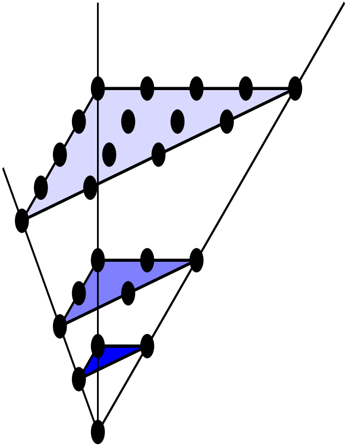
- Use the techniques in surveys: Stanley 1989, Brenti 1993, Brändén 2016

(Not always clear how to apply these...)

Two main philosophies arise for proving unimodality of $h^*(P; z)$:

1. Decompose P and apply algebraic results.
2. Recursions and real-rootedness.

Lattice polytopes associate naturally to **semigroup algebras**.



- $\text{cone}(P) := \text{span}_{\mathbb{R}_{\geq 0}}\{(p,1) : p \in P\} \subset \mathbb{R}^{n+1}$.
- For $v := (v_1, \ldots, v_{n+1}) \in \mathbb{Z}^{n+1}$ define a monomial
$$x^v := x_1^{v_1} \cdots x_{n+1}^{v_{n+1}}.$$
- $\mathbb{C}[P] := \mathbb{C}[x^v : v \in \text{cone}(P)]$.
- With the grading
$$\deg(x^v) := v_{n+1},$$
$\mathbb{C}[P]$ is a graded semigroup algebra sometimes called the **Ehrhart ring of** P.
- $\dfrac{h^*(P;z)}{(1-z)^{d+1}}$ = the **Hilbert series of** $\mathbb{C}[P]$.

Algebraic Properties of $\mathbb{C}[P]$:

- $\mathbb{C}[P]$'s are examples of **Cohen-Macaulay integral domains**. [Hochster, 1972]
- Consequently, many conjectures on Ehrhart unimodality are related to algebraic properties of $\mathbb{C}[P]$.

P is called **IDP** or has the **Integer Decomposition Property** if for every $t \in \mathbb{Z}_{>0}$ and every $v \in tP \cap \mathbb{Z}^n$ there exist $v^{(1)}, \ldots, v^{(t)} \in P \cap \mathbb{Z}^n$ such that

$$v = v^{(1)} + \cdots + v^{(t)}.$$

- i.e. $\mathbb{C}[P]$ is **integrally closed**.

P is called **Gorenstein** if $h^*(P;z)$ is symmetric.

- i.e. if $\deg(h^*(P;z)) = s$ then $h^*_i = h^*_{s-i}$ for all $i = 0, 1, \ldots, s$.
- i.e. $\mathbb{C}[P]$ is a **Gorenstein ring**. [Stanley, 1978]
- If $\deg(h^*(P;z)) = n$ then P is called **reflexive**.

Two Major Open Problems:

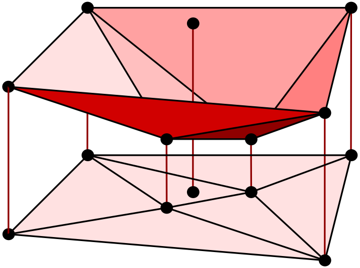
Conjecture (Hibi, Ohsugi, 1992). If P is Gorenstein and IDP then $h^*(P; z)$ is unimodal.

- Special case of an algebraic conjecture of Stanley (1989) about standard graded Gorenstein integral domains.

Question (Schepers, Van Langenhoven, 2013). If P is IDP, is it true that $h^*(P; z)$ is always unimodal?

The Major Positive Result:

- A **triangulation** T of P into lattice simplices is called:
  - **regular** if it is the projection of the lower hull of a lifting of the lattice points in P into $\mathbb{R}^{n+1}$.
  - **unimodular** if all simplices $\sigma \in T$ have unit volume (i.e. $h^*(\sigma;1) = 1$).



#photocred [*Triangulations*; De Loera, Rambau, Santos, 2010]

- P has a **regular unimodular triangulation** $\Rightarrow$ P is IDP.

Theorem (Bruns, Römer, 2007). If P is Gorenstein and admits a regular unimodular triangulation then $h^*(P;z)$ is unimodal.

Theorem (Bruns, Römer, 2007). If P is Gorenstein and admits a regular unimodular triangulation then $h^*(P; z)$ is unimodal.

- Applied to a wide variety of polytopes to recover Ehrhart unimodality results
- **Regular unimodular triangulations and/or identification of Gorenstein:**
  - order polytopes [Stanley, 1972]
  - double poset polytopes [Chappell, Friedl, Sanyal, 2016]
  - twinned poset polytopes [Hibi, Matsuda, Tsuchiya, 2015]
  - (n,k)-hypersimplices [Stanley, 1977; Sturmfels, 1996]
  - r-stable (n, k)-hypersimplices [Braun, LS, 2014]
  - positroid polytopes [Ardila, Rincón, Williams, 2015]
  - alcoved polytopes [Lam and Postnikov, 2007]
  - s-lecture hall simplices [Hibi, Olsen, Tsuchiya, 2016] [Beck, Braun, Köppe, Savage, Zafeirakopoulos, 2016] [Brändén, LS, 2017]
  - s-lecture hall order polytopes [Brändén, Leander, 2016]
  - etc...

Box Polynomials and Box Unimodality:

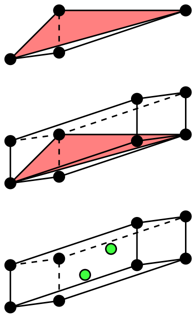
- $\Delta := \mathrm{conv}(v^{(1)}, \ldots, v^{(d)}, v^{(d+1)}) \subset \mathbb{R}^n$ a simplex.

- The **box polynomial of** Δ is

$$B(\Delta; z) := \sum_{v \in \Pi^\circ(\Delta) \cap \mathbb{Z}^{n+1}} z^{v_{n+1}},$$

where the **open fundamental parallelpiped** of Δ is

$$\Pi^\circ(\Delta) := \left\{ \sum_{i=1}^{d+1} \lambda_i (v^{(i)}, 1) : 0 < \lambda_i < 1 \right\}.$$



Theorem (Betke, McMullen, 1985). Fix a triangulation T of the boundary of a reflexive polytope P. Then

$$h^*(P; z) = \sum_{\Delta \in T} h(\mathrm{link}(\Delta); z) B(\Delta; z),$$

where $h(\mathrm{link}(\Delta); z)$ denotes the h-polynomial of the link of Δ in T.

Box Polynomials and Box Unimodality:

Definition (Schepers and Van Langenhoven, 2013). A regular triangulation T of the boundary of an n-dimensional polytope P is called **box unimodal** if $B(\Delta; z)$ is unimodal for all $\Delta \in T$.

- If P is reflexive and has a box unimodal triangulation (with box polynomials of appropriate degrees...) then $h^*(P; z)$ is unimodal.
- **Question (Schepers, Van Langenhoven, 2013).** Does the boundary of every IDP reflexive lattice polytope admit a box unimodal triangulation?
- **Question (Braun, 2016).** Which lattice simplices have unimodal box polynomials?

Recursions and Real-rootedness:

An increasingly popular technique for proving Ehrhart unimodality is to show that all roots of $h^*(P; z)$ are real numbers.

Lemma. Suppose

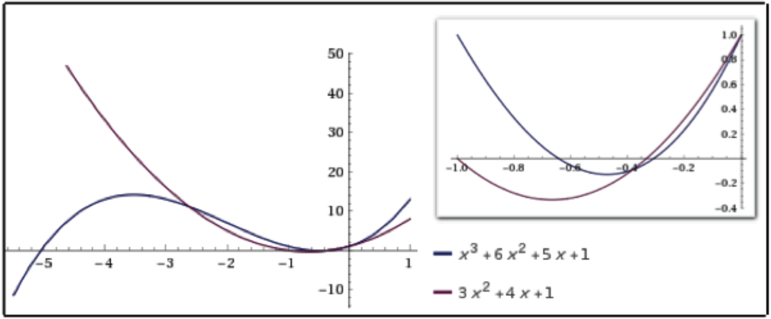
$$p(z) = a_0 + a_1 z + \cdots + a_d z^d$$

satisfies $a_0, \ldots, a_d \in \mathbb{R}_{\geq 0}$.

1. If $p(z)$ has only real-roots then it is log-concave, i.e., $a_i^2 \geq a_{i-1} a_{i+1}$ for all i.
2. If $p(z)$ is log-concave and a_i are all positive the $p(z)$ is unimodal.

The key to proving real-rootedness:

- Identify **recursions**.
- Show recursions preserve **interlacing** of real-roots.



f **interlaces** g, denoted $f \preceq g$.

A sequence of real-rooted polynomials

$$f_1 \preceq f_2 \preceq \cdots \preceq f_m$$

is called **interlacing** if $f_i \preceq f_j$ for all $1 \leq i < j \leq m$.

To prove real-rootedness we search for recursions for our polynomials that can be stated using **interlacing preservers**.

This red $m \times k$ matrix of polynomials is an **interlacing preserver**:

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ z & 1 & 1 & & \vdots \\ z & z & 1 & \ddots & 1 \\ \vdots & & \ddots & \ddots & 1 \\ z & z & \cdots & z & \ddots \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_k \end{pmatrix} = \begin{pmatrix} g_0 \\ g_1 \\ \vdots \\ g_m \end{pmatrix}$$

$$f_1 \preceq f_2 \preceq \cdots \preceq f_k \qquad \Rightarrow \qquad g_1 \preceq g_2 \preceq \cdots \preceq g_m$$

- **The following have real-rooted h^*-polynomials:**
  - s-lecture hall polytopes [Savage, Visontai, 2014]
  - Zonotopes [Beck, Jochemko, McCullough, 2016]
  - (Sufficiently) dilated lattice polytopes [Jochemko, 2016]
  - Some order polytopes [Wagner, 1992]
  - Some r-stable hypersimplices [Braun, LS, 2014]
  - Some symmetric edge polytopes [Higashitani, Kummer, Michałek, 2016]
  - Some simplices for numeral systems [LS, 2017]

Key Observations so far:

1. Popular techniques for proving Ehrhart unimodality:
   (i) Prove Gorenstein and existence of regular unimodular triangulation.
   (ii) Prove box unimodality.
   (iii) Prove real-rootedness.

2. Note that (ii) is less popular... Perhaps not well-understood?

3. Oftentimes, if (i) is easy then (iii) is hard, or vice-versa:
   - **For r-stable (n, k)-hypersimplices:**
     - Existence of regular unimodular triangulations [Braun, LS, 2014]
     - Characterization of Gorenstein [Hibi, LS, 2014]
     - Few known to be real-rooted [Braun, LS, 2014]
   - **For s-lecture hall simplices:**
     - All real-rooted [Savage, Visontai, 2014]
     - Partial results on Gorenstein [Hibi, Olsen, Tsuchiya, 2016]
     - Few known to have regular unimodular triangulations [Hibi, Olsen, Tsuchiya, 2016] [Beck, Braun, Köppe, Savage, Zafeirakopoulos, 2016] [Brändén, LS, 2017]

- (i) and (ii) have strong geometric ties to P.
- (iii) is increasingly popular, but removes real-rootedness proof to recursion (independent of geometry?).

Question. Can we better understand the geometric underpinnings of **Ehrhart real-rootedness**?

Where to start?

Simplices are hard enough:

Benefits:

- Simple combinatorial structure (i.e. Boolean face lattice).
- Easy-to-work-with interpretation of h^*-polynomials.

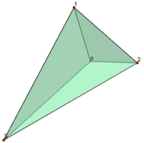
For large families of simplices still challenging to characterize:

- IDP
- Gorenstein
- Existence of Regular Unimodular Triangulations
- Box Polynomials
- Real-rootedness

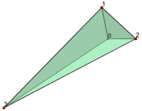
Focus on simplices of the form:

$$\Delta_{(1,q)} := \text{conv}(e_1, \ldots, e_n, -q) \subset \mathbb{R}^n,$$

where $e_1, \ldots, e_n$ are the standard basis vectors and $q := (q_1, \ldots, q_n)$ is a sequence of weakly increasing positive integers.



$q = (1, 1, 1)$



$q = (1, 2, 3)$

Features:

- Toric varieties are weighted projective spaces
- Reflexivity is characterized [Conrads, 2002]
- Reflexivity + IDP is characterized [Braun, Davis, LS, 2016]
- Counterexamples to Ehrhart unimodality conjectures [Payne, 2008]
- $h^*(\Delta_{(1,q)}; z)$ has arithmetic formula in terms of q [Braun, Davis, LS, 2016]

Simplices for Numeral Systems:

Question. What do $\Delta_{(1,q)}$ with real-rooted h^*-polynomials look like?

Approach:

- $\mathcal{Q} :=$ collection of all $\Delta_{(1,q)}$.
- Stratify $\mathcal{Q}$ by **normalized volume**.
- Recursions evolve when normalized volumes associated to place values in **positional numeral systems**.

Proposition (Nill, 2007). The normalized volume of $\Delta_{(1,q)}$ is

$$1 + q_1 + q_2 + \cdots + q_n.$$

Proposition (Braun, Davis, LS, 2017). The h^*-polynomial of $\Delta_{(1,q)}$ is

$$h^*(\Delta_{(1,q)}; z) = \sum_{b=0}^{q_1+q_2+\cdots+q_n} z^{\omega(b)},$$

where

$$\omega(b) = b - \sum_{i=1}^{n} \left\lfloor \frac{q_i b}{1 + q_1 + q_2 + \cdots + q_n} \right\rfloor .$$

Positional Numeral Systems:

- A **numeral system** is a sequence of positive integers (*place values*)

$$a = (a_n)_{n=0}^{\infty} \qquad \text{satisfying} \qquad a_0 := 1 < a_1 < a_2 < \cdots .$$

$$a = (2^n)_{n=0}^{\infty} = (1, 2, 4, 8, 16, \ldots, 2^n, \ldots)$$

$$102 = 1 \cdot 2^6 + 1 \cdot 2^5 + 0 \cdot 2^4 + 0 \cdot 2^3 + 1 \cdot 2^2 + 1 \cdot 2^1 + 0 \cdot 2^0$$

- a **numeral** is our representation of a number with **digits**:
  - the binary (base 2) representation of 102 is the numeral

$$\eta = 1100110.$$

Main Idea. By associating simplices $\Delta_{(1,q)}$ for $q \in \mathbb{R}^n$ with normalized volume a_n to a numeral system $(a_n)_{n=0}^{\infty}$, we can study the combinatorics of $h^*(\Delta_{(1,q)}; z)$ recursively in terms of the numerals η w.r.t. to a.

Example: The Binary System.

- Let $a = (2^n)_{n=0}^{\infty} = (1, 2, 4, 8, 16, \ldots, 2^n, \ldots)$ be the binary numeral system.
- For each n let $q := (1, 2, 4, \ldots, 2^{n-1})$.
- Then $h^*(\Delta_{(1,q)}; 1) = 1 + 1 + 2 + 4 + \cdots + 2^{n-1} = 2^n = a_n$.
- Recall

$$h^*(\Delta_{(1,q)}; 1) = \sum_{b=0}^{q_1+q_2+\cdots+q_n} z^{\omega(b)},$$

where

$$\omega(b) = b - \sum_{i=1}^{n} \left\lfloor \frac{q_i b}{1 + q_1 + q_2 + \cdots + q_n} \right\rfloor.$$

- Apply some inductive reasoning...
- Discover that $\omega(b) = \#$ of 1's in base 2 representation of $b := \mathrm{supp}_2(b)$

Theorem (LS, 2017).

$$h^*(\Delta_{(1,q)}; z) = \sum_{b=0}^{2^n-1} z^{\mathrm{supp}_2(b)} = (1+z)^n.$$

Another Example: The Factoradics.

- $a = ((n+1)!)_{n=0}^{\infty} = (1, 2, 6, 24, \ldots)$ is the **factoradic** numeral system.
- The factoradic representation of $0 \leq b < n!$ is the **Lehmer Code** of $\pi^{(b)}$, the b^{th} lexicographically largest permutation in S_n.
- Define the generating polynomial

$$B_n(z) := \sum_{\pi \in S_n} z^{\mathrm{maxDes}(\pi)},$$

  where $\mathrm{maxDes}(\pi) = 0$ if $\mathrm{Des}(\pi) = \emptyset$.

- Let $q = \big([z].B_{n+1}(z), [z^2].B_{n+1}(z), \ldots, [z^n].B_{n+1}(z)\big)$....
- Discover that $\omega(b) = \mathrm{des}(\pi^{(b)})$....

Theorem (LS, 2017).

$$h^*(\Delta_{(1,q)}; z) = \sum_{b=0}^{(n+1)!-1} z^{\mathrm{des}(\pi^{(b)})} = A_{n+1}(z).$$

- By stratifying $\mathcal{Q}$ by normalized volumes associated to numeral systems we are recovering classic families of real-rooted polynomials!
- The examples so far are called **reflexive systems** since all h^*-polynomials are symmetric.
- If we drop the symmetry requirement, we obtain larger families of simplices with real-rooted h^*-polynomials.
- These have intriguing connections to **box polynomials**....

More Examples: The Base-r Numeral Systems:

- The **base-r numeral system** is $a = (r^n)_{n=0}^{\infty}$.

- Here, we let

$$q = ((r-1), (r-1)r, (r-1)r^2, \ldots, (r-1)r^{n-1}),$$

since then

$$h^*(\Delta_{(1,q)}; 1) = 1 + \sum_{k=0}^{n-1} (r-1)r^k = r^n = a_n.$$

- Let $\mathcal{B}_{(r,n)} := \Delta_{(1,q)}$ be the n^{th} **base-r simplex**.

- For $r \geq 2$ and $n \geq 1$ we let

$$f_{(r,n)} := (1 + z + z^2 + \cdots + z^{r-1})^n.$$

More Examples: The Base-r Numeral Systems:

- $r = 4$ and $n = 2$:

$$f_{(r,n)} = 1 + 2z + 3z^2 + 4z^3 + 3z^4 + 2z^5 + z^6.$$

$$f_{(r,n)} = 1z^{0\cdot(r-1)+0} + 2z^{0\cdot(r-1)+1} + 3z^{0\cdot(r-1)+2} + 4z^{1\cdot(r-1)+0} + 3z^{1\cdot(r-1)+1} + 2z^{1\cdot(r-1)+2} + 1z^{2\cdot(r-1)+0}.$$

- $f^{(2)}_{(r,n)} = 3 + 2z, \quad f^{(1)}_{(r,n)} = 2 + 3z, \quad f^{(0)}_{(r,n)} = 1 + 4z + 1z^2.$

Theorem (LS, 2017). We have the interlacing sequence

$$f^{(r-2)}_{(r,n)} \prec f^{(r-3)}_{(r,n)} \prec \cdots \prec f^{(1)}_{(r,n)} \prec f^{(0)}_{(r,n)}.$$

Moreover,

$$h^*(\mathcal{B}_{(r,n)}; z) = f^{(0)}_{(r,n)} + z \sum_{\ell=1}^{r-2} f^{(\ell)}_{(r,n)}$$

Corollary (LS, 2017). $h^*(\mathcal{B}_{(r,n)}; z)$ are real-rooted.

Connections to Box Polynomials:

$$h^*(\mathcal{B}_{(r,n)}; z) = f^{(0)}_{(r,n)} + z \sum_{\ell=1}^{r-2} f^{(\ell)}_{(r,n)}$$

$$\Downarrow$$

$$h^*(\mathcal{B}_{(r,n)}; z) = a(z) + zb(z) ...$$

Theorem (Stapledon ?). Let P is a lattice polytope containing an interior lattice point. There exist unique polynomials $a(z)$ and $b(z)$ such that

$$h^*(P; z) = a(z) + zb(z),$$

where $a(z) = z^d a\left(\frac{1}{z}\right)$ and $b(z) = z^{d-1} b\left(\frac{1}{z}\right)$.

Since $\mathcal{B}_{(r,n)}$ is a simplex, we can express these polynomials simply as:

$$a(z) = \sum_{\Delta \in \mathcal{B}_{(r,n)}} (1 + z + \cdots + z^{n-\dim(\Delta)-1}) B(\Delta; z), \quad \text{and}$$

$$b(z) = \frac{1}{z} \sum_{\Delta \in \mathcal{B}_{(r,n)}} (1 + z + \cdots + z^{n-\dim(\Delta)-1}) B(\operatorname{conv}(\Delta, \mathbf{0}); z).$$

So perhaps we should revisit box polynomials for simplices....

In Summary:

- Ehrhart unimodality is a rich and challenging area of research!
- Ehrhart unimodality results center around two central ideas:
  - decompose and apply algebraic results.
  - recursions and real-rootedness
- The applicability and usefulness of these techniques is still not completely understood, not even for "popular polytopes" or "simple families."
  - i.e. those polytopes used as examples in this talk.
  - i.e. large families of simplices.
- The relationship and disparity between applicability of approaches (i) and (ii) is not so clear.
  - one is often easier than the other
  - can we better understand their relationship in the case of simplices?

Things to do:

- Answer the conjecture of Hibi and Ohsugi!
- Answer the question of Schepers and Van Langenhoven!
- Answer the question of Braun!
  - I.e. better understand unimodality of box polynomials for simplices.
- Work on popular examples!
  - Help characterize better the unimodality and applicability of these results for the families of polytopes mentioned here!
  - The applicability of techniques (i) and (ii) is only characterized for a few of the examples we discussed today!
- Get creative!
  - Develop new families of polytopes for which to test theories in Ehrhart unimodality.

Thank You!