Day 2 talk 4

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"On matrix-valued log-concave functions"

Note: Speaker will provide notes/write up on arxiv in a few weeks.

<u>Motivation</u>

Theorem (Raufi '13): Let $n_0, n_1, d \geq 1$ and

Let $g : \mathbb{R}^{n_0+n_1} \longrightarrow M_d^+ \leftarrow$ $d \times d$ symmetric matrices $> 0$

For $t \in \mathbb{R}^{n_0}$ define

$$\alpha(t) = \int_{\mathbb{R}^{n_1}} g(t,y)\,dy$$

<u>If</u> $g$ is N-log-concave, then $\alpha$ is N-log-concave.

Ex. When $d=1 \rightarrow g : \mathbb{R}^{n_0+n_1} \longrightarrow \mathbb{R}_+ (= M_1^+)$ log-concave $\Rightarrow \alpha$-log-concave

<u>i.e.</u> Prekopa

Q: What can be log-concavity when $d > 1$?

Related question in the complex case (complex interpolation) $n=1$

<u>Ex.</u>

$\mathbb{D} \subset \mathbb{R}^2 = \mathbb{C}$

$\{g(\theta)\}_{\theta \in \partial \mathbb{D}}$ $g(\theta)$ Hilbert structure on $\mathbb{C}^d$

- introduce/try to solve $\Theta^g(z) = \partial_{\bar z}\left(g(z)^{-1}\partial_z g\right) = 0$

- extremal ~~condition~~ for $\underbrace{g(z)\Theta^g(z) \leq 0}$ corresponds to log-concavity

When $d=1$ $g(z) = e^{-u(z)}$

$\Theta^g = -\Delta u(z)$

$\rightsquigarrow e^{-u(z)}\Delta u(z) \geq 0$

subharmonicity

$n, d > 1$ known in the complex case

(based on vector bundles)

Notation : ~~[illegible]~~ – $g : \mathbb{R}^n \to M_d^+$ $C^2$-function

for $j, k \le n$, define $\Theta^g_{j,k} = \Theta^g_{j,k}(x) = \partial_{x_k}(g(x)^{-1}\partial_{x_j} g)$

$$= g^{-1}\partial_{x_k}\partial_{x_j} g - g^{-1}\partial_{x_k} g\, g^{-1}\partial_{x_j} g \in M_d$$

– Given Euclidean space

$(E, \langle , \rangle, \|\cdot\|)$ and $g \in L(E)$ sym $> 0$

denote $\langle v, v\rangle_g = \langle gv, v\rangle = \|v\|_g$ .

(Fix $x$ & ) Here $E = \mathbb{R}^d$ or $M_{dn}$

for $u, v \in \mathbb{R}^d$

$$\langle \Theta^g_{j,k} u, v\rangle_{g(x)} = \langle \partial^2_{x_k x_j} g\, u, v\rangle - \langle g^{-1}\partial_{x_k} g u, g^{-1}\partial_{x_j} g v\rangle$$

$$= \langle u, \Theta^g_{k,j} v\rangle$$

– Introduce $\Theta^g = \Theta^g(x) = [\Theta^g_{j,k}]_{j,k \le n} \in M_n \otimes M_d$

will be seen as an ~~element of the~~ operator on $M_{dn}$

as follows. For $U, V \in M_{dn}$ $U_i, V_i \in \mathbb{R}^d$

$U = [U_1, \ldots, U_n]$, $V = [V_1 \ldots V_n]$

Side remark

$\langle U, V\rangle_{I_n \otimes g} = tr((gU)^T V) = \sum_{k=1}^n \langle g U_k, V_k\rangle$

subscript $[gU_1 \ldots gU_n]$

$$\Theta^g U = \Big[ \underbrace{\sum_{j=1}^{n} \Theta^g_{j,k} u_j}_{\in \mathbb{R}^d} \Big]_{k=1\dots n}$$

$\Theta^g$ as an operator on $M_{d,n}$

<u>Def.</u> $g : \mathbb{R}^n \to M_d^+$ is N-log-concave if at every $x \in \mathbb{R}^n$ $\Theta^g(x)$ is a sym $\leq 0$ operator on $M_{dn}$ ; i.e.

$\forall\, U = [u_1, \dots, u_n] \in M_{dn}$ $\quad \langle \Theta^g U, U \rangle \leq 0$

$$\overset{\shortparallel}{\sum_{k,j=1}^{n}} \langle \Theta^g_{jk} u_j, u_k \rangle$$

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<u>Example</u> $d=1$ $\quad \Theta^g \in L(\mathbb{R}^n)$ $\quad$ (lin. map on $\mathbb{R}^n$)

$\Theta^g = \text{Hess}(\log g)$

$\underbrace{g(x)}_{>0}\ \text{Hess}(\log g) \leq 0$

$\rightsquigarrow$ classically log-concave

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Reference Book: ______________ chapter 7

J.P. Demailly

- complex case, many examples
- otherwise fewer

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Ex $n, d > 1$ $\quad n = d = 2$ $\quad g : \mathbb{R}^2 \to M_2^+ \subset \mathbb{R}^3$

$$g(x) = g(x_1, x_2) = \mathrm{Id}_2 - \begin{pmatrix} x_1 + s x_2^2 & x_1 x_2 \\ x_1 x_2 & x_2^2 + s x_1^2 \end{pmatrix}$$

$$\text{Spect}\,[g\, \Theta^g_{jk}(0)] = \{-3, -1, -1-2s, 1-2s\}$$

For $s > \frac{1}{2}$ $\;< 0$ $\rightsquigarrow$ in a neighborhood of 0 is log-concave

Proof of Rawi (ideas):

- Do a classical proof.

$g : \mathbb{R}^{n+1} \to \mathbb{R}^+$ $\qquad$ $n_0 = 1$, $g = e^{-\varphi}$ $\quad \varphi$ convex

check that $\alpha(t) = \int_{\mathbb{R}^n} e^{-\varphi(t,y)}\,dy$ is log-concave

<u>1st step</u> Computation:

$$\partial^2_{tt}(\log\alpha) = \frac{\partial^2_{tt}\alpha}{\alpha} - \left(\frac{\partial_t \alpha}{\alpha}\right)^2 \qquad \text{(want a certain sign)}$$

$$= -\int_{\mathbb{R}^n} \partial_{tt}\varphi\, e^{-\varphi} + \mathrm{Var}_\mu(\partial_t \varphi) \qquad d\mu = \frac{e^{-\varphi(t,y)}}{\int_{\mathbb{R}^n} e^{-\varphi}}\,dy$$

<u>2nd step</u> Relate derivative in $t$ and in $y$

$$\det \mathrm{Hess}_{(t,y)}\varphi = \det \left(\begin{array}{c|c} \partial^2_{tt}\varphi & (\nabla\partial_t\varphi)^T \\ \hline \nabla\partial_t\varphi & \mathrm{Hess}_y\varphi \end{array}\right)$$

$$\partial^2_{tt}\varphi \geq (\mathrm{Hess}_y\varphi)^{-1}\nabla\partial_t\varphi \cdot \nabla\partial_t\varphi$$

$=$ HRMA

<u>3rd step</u>: Spectral step

B.-L. $\quad$ For $\varphi : \mathbb{R}^n \to \mathbb{R}^+$ convex then $\forall f \in L^2(e^{-\varphi})$, $C^1$, we have

$f = \partial_t\varphi$ $\quad$ $\mathrm{Var}_\mu(f) \leq \int_{\mathbb{R}^n} (\mathrm{Hess}\,\varphi)^{-1}\nabla f \cdot \nabla f\, d\mu$

$$d\mu = \frac{e^{-\varphi}}{\int e^{-\varphi}}\,dy$$

# Brascamp-Lieb variance inequality for matrix weights

$g : \mathbb{R}^n \to M_d^+$ $C^2$-smooth

$A_n$: $F : \mathbb{R}^n \to \mathbb{R}^d$ belongs to $L^2(g)$ if

$$\int_{\mathbb{R}^n} \|F\|_g^2 < \infty$$

$$= \int_{\mathbb{R}^n} \langle g(x) F(x), F(x) \rangle \, dx$$

Well-defined if $\int_{\mathbb{R}^n} |g| < \infty$, $F \in L^2$,

Notation:

$$\nabla F = \nabla F(x) = [\partial_1 F, \ldots, \partial_n F] \in M_{dn}$$

$\boxed{\text{Theorem}}$ Let $g : \mathbb{R}^n \to M_d^+$ $C^2$ N-log-concave.

Assume that $\int |g| < \infty$ and set $Z := \int g \in M_d^+$.

Then, for any $F : \mathbb{R}^n \to \mathbb{R}^d$ in $L^2(g) \cap C^1$ we have

$$\int \| F(x) - Z^{-1} \int g F \|_{g(x)}^2 \, dx$$

$$\leq \int \langle -(\Theta^g(x))^{-1} \nabla F, \nabla F \rangle_{Id_n \otimes g} \, dx$$

Ideas from
Prior result and proof by Hörmander.

Proof sketch: introduce $\mathcal{L} F = \Delta F = \sum_{j=1}^{n} (g(x)^{-1} \partial_j g) \partial_j F$

Facts ① $F, G \in C^1$ compactly supported

$$\int_{\mathbb{R}^n} \langle \mathcal{L} F, H \rangle_g = -\int \langle \nabla F, \nabla H \rangle_{Id_n \otimes g}$$

Fact 2. Bochner formula

$$\int \|\mathcal{Z} F\|_g^2 = \int \langle -\Theta^g \nabla F, \nabla F \rangle_{\mathrm{Id}_n \otimes g} + \int \sum_{j,k=1}^{n} \|\partial_{jk}^2 F\|_g^2$$

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Proof of Prekopa-Rauti

$V_0 \in [v_1 \dots v_n] \in M_{n_0, d}$. At every $t_0 \in \mathbb{R}^{n_0}$

$$\langle \Theta^\alpha v_0, v_0 \rangle_\alpha \leq 0$$

$$\| $$

$$\int_{\mathbb{R}^{n_1}} \langle \Theta_{00}^g V_0, V_0 \rangle_{\mathrm{Id}_n \otimes g} \, dy - \int \| F - \alpha(t_0)^{-1} \int g F \|_{g(t_0, y)}^2 \, dy$$

$$F = \underbrace{\sum_{j=1}^{n_0} (g^{-1} \partial_{x_j} g) v_j}_{} \quad (g = e^{-\varphi})$$

$$\boxed{\nabla_y F = \Theta_{10}^g V_0}$$

$$\Theta^g = \left[ \begin{array}{c|c} \Theta_{00}^g & \Theta_{01}^g \\ \hline \Theta_{10}^g & \Theta_{11}^g \end{array} \right]$$

$\Theta_{0,0}^g \in L(M_{n_0, d})$

$\Theta_{1,0}^g \in L(M_{n_0, d}, M_{n_1, d})$

$\Theta_{11}^g \in L(M_{n_1, d})$

<u>Prop.</u> at any $(t, y)$ and for any $V_0 = [v_1, \dots, v_n]$

$$\langle (-\Theta_{00}^g) V_0, V_0 \rangle_{\mathrm{Id}_{n_0} \otimes g} \geq \langle (-\Theta_{11}^g)^{-1} \Theta_{10}^g V_0, \Theta_{10}^g V_0 \rangle_{\mathrm{Id}_{n_1} \otimes g}$$

generalized Monge-Ampere equation for matrix valued weights.