

Day 2 talk 4

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"On matrix-valued log-concave functions"

Note: Speaker will provide notes / write up on arXiv in a few weeks.

Motivation

Theorem (Rauzy '13): Let $n_0, n_1, d \geq 1$ and

Let $g: \mathbb{R}^{n_0+n_1} \rightarrow M_d^+$ $d \times d$ symmetric matrices ≥ 0

For $t \in \mathbb{R}^{n_0}$ define

$$\alpha(t) = \int_{\mathbb{R}^{n_1}} g(t, y) dy$$

If g is N -log-concave, then α is N -log-concave.

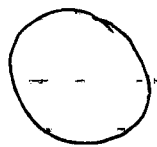
Ex. When $d=1 \rightarrow g: \mathbb{R}^{n_0+n_1} \rightarrow \mathbb{R}_+ (= M_1^+)$ log-concave $\Rightarrow \alpha$ -log-concave

i.e. Prékopa

Q: What can be log-concavity when $d > 1$?

Related question in the complex case (complex interpolation) $n=1$

Ex.



$$\bar{D} \subset \mathbb{R}^2 = \mathbb{C}$$

$$\{g(\theta)\}_{\theta \in \bar{D}}$$

$g(\theta)$ Hilbert structure on \mathbb{C}^d

introduce / try to solve $\theta^g(z) = \delta_z^g (g(z)^{-1} dz g) = 0$

Extremal condition for $g(z) \theta^g(z) \leq 0$

When

$$d=1 \quad g(z) = e^{-u(z)} \\ \theta^g = -\Delta u(z)$$

$$\Rightarrow e^{-u(z)} \Delta u(z) \geq 0 \\ \text{subharmonicity}$$

Corresponds to log-concavity

$n, d > 1$ known - in the complex case

(based on vector bundles)

Notation: ~~Define~~ $\bar{g}: \mathbb{R}^n \rightarrow M_d^+$ C^2 -function

for $j, k \leq n$, define $\Theta_{j,k}^{\bar{g}} = \Theta_{j,k}^{\bar{g}}(x) = d_{x_k}(g(x)^{-1} d_{x_j} g)$

$$= g^{-1} d_{x_k} d_{x_j} g - \cancel{g^{-1} d_{x_k} g g^{-1} d_{x_j} g}$$

$$g^{-1} d_{x_k} g g^{-1} d_{x_j} g \in M_d$$

- Given Euclidean space

$(\tilde{E}, \langle \cdot, \cdot \rangle, \|\cdot\|)$ and $\bar{g} \in L(E)$ sym > 0

denote $\langle v, v \rangle_{\bar{g}} = \langle \bar{g}v, v \rangle = \|v\|_{\bar{g}}$

Here $E = \mathbb{R}^d$ or M_d

$$\begin{aligned} \langle \Theta_{j,k}^{\bar{g}} u, v \rangle_{g(x)} &= \langle d_{x_k}^2 g^j u, v \rangle - \langle g^{-1} d_{x_k} g u, g^{-1} d_{x_j} g v \rangle \\ &= \langle u, \Theta_{k,j}^{\bar{g}} v \rangle \end{aligned}$$

- Introduce $\Theta^{\bar{g}} = \Theta^{\bar{g}}(x) = [\Theta_{j,k}^{\bar{g}}]_{j,k \leq n} \in M_n \otimes M_d$

will be seen as an operator on M_d ~~element of M_d~~

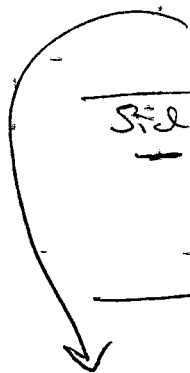
as follows. For $u, v \in M_d$ $u_i, v_i \in \mathbb{R}^d$

$$u = [u_1, \dots, u_n], v = [v_1, \dots, v_n]$$

$$\langle u, v \rangle_{\Theta^{\bar{g}}} = \text{tr}((\bar{g}u)^T v) = \sum_{k=1}^n \langle \bar{g}u_k, v_k \rangle$$

subscript $[g_{u_1} \dots g_{u_n}]$

Side remark



$$\Theta^g u = \left[\underbrace{\sum_{j=1}^n \Theta_{j,k}^g u_j}_{\in \mathbb{R}^d} \right]_{k=1, \dots, n}$$

Θ^g as an operator on $M_{d,n}$

Def. $g: \mathbb{R}^n \rightarrow M_d^+$ is N -log-concave ^{iff} at every $x \in \mathbb{R}^n$ $\Theta^g(x)$ is a

sym ≤ 0 operator on $M_{d,n}$ i.e.

$$\forall U = [u_1, \dots, u_n] \in M_{d,n} \quad \langle \Theta^g U, U \rangle \leq 0$$

$$\Leftrightarrow \sum_{k,j=1}^n \langle \Theta_{j,k}^g u_j, u_k \rangle$$

Example $d=1$ $\Theta^g \in L(\mathbb{R}^n)$ (lin. map on \mathbb{R}^n)

$$\Theta^g = \text{Hess } g(\log g)$$

$$g(x) > 0 \quad \text{Hess } (\log g) \leq 0$$

\leadsto classically log-concave

Reference Book:

chapter 7

J.P. Demailly

- complex case, many examples
- otherwise fewer

Ex $n, d > 1$ $n=d=2$ $g: \mathbb{R}^2 \rightarrow M_2^+ \subset \mathbb{R}^3$

$$g(x) = g(x_1, x_2) = \text{Id}_2 - \begin{pmatrix} x_1 + 5x_2^2 & x_1 x_2 \\ x_1 x_2 & x_2^2 + 5x_1^2 \end{pmatrix}$$

$$\text{Spect } [g \Theta_{j,k}^g(0)] = \{-3, -1, -1-2s, 1-2s\}$$

For $s > \frac{1}{2}$ $\text{for } < 0 \leadsto$ in a neighborhood of 0 is log-concave

Proof of Rarfi (ideas):

- Do a classical proof.

$$g: \mathbb{R}^{n+1} \rightarrow \mathbb{R}^+$$

$$n_0 = 1$$

$$g = e^{-\varphi}$$

φ convex

check that

$$\alpha(t) = \int_{\mathbb{R}^n} e^{-\varphi(t,y)} dy \text{ is log-concave}$$

1st step

Computation:

$$\Delta_{tt}^2 (\log \alpha) = \frac{\Delta_{tt}^2 \alpha}{\alpha} - \left(\frac{\Delta_t \alpha}{\alpha} \right)^2 \quad (\text{want a certain sign})$$

$$= - \int_{\mathbb{R}^n} \Delta_{tt} \varphi e^{-\varphi} + \text{Var}_{\mu}(\Delta_t \varphi) \quad d\mu = \frac{e^{-\varphi(t,y)}}{\int_{\mathbb{R}^n} e^{-\varphi}} dy$$

2nd step

Relate derivative in t and in y

$$\det \text{Hess}_{(t,y)} \varphi = \det \begin{pmatrix} \Delta_{tt}^2 \varphi & (\nabla \Delta_t \varphi)^T \\ \nabla \Delta_t \varphi & \text{Hess}_y \varphi \end{pmatrix}$$

$$\Delta_{tt}^2 \varphi \geq (\text{Hess}_y \varphi)^{-1} \nabla \Delta_t \varphi \cdot \nabla \Delta_t \varphi$$

= HRTA

3rd step

Spectral step

B-L. For $\varphi: \mathbb{R}^n \rightarrow \mathbb{R}^+$ convex then

$\forall f \in L^2(e^{-\varphi})$, e^{φ} we have

$$f = \Delta_t \varphi$$

$$\text{Var}_{\mu}(f) \leq \int_{\mathbb{R}^n} (\text{Hess}_y \varphi)^{-1} \nabla f \cdot \nabla f d\mu$$

$$d\mu = \frac{e^{-\varphi}}{\int e^{-\varphi}} dy$$

Brascamp-Lieb variance inequality for matrix weights

$$g: \mathbb{R}^n \rightarrow M_d^+ \quad C^2\text{-smooth.}$$

A_n : $F: \mathbb{R}^n \rightarrow \mathbb{R}^d$ belongs to $L^2(g)$ if

$$\int_{\mathbb{R}^n} \|F\|_g^2 < \infty$$

$$= \int_{\mathbb{R}^n} \langle g(x) F(x), F(x) \rangle dx$$

well-defined if $\int_{\mathbb{R}^n} |g| < \infty$, $F \in L^2$,

Notation:

$$\nabla F = \nabla F(x) = [\partial_1 F, \dots, \partial_n F] \in M_{d \times n}$$

Theorem

Let $g: \mathbb{R}^n \rightarrow M_d^+$ C^2 N -log-concave.

Assume that $\int |g| < \infty$ and set $Z := \int g \in M_d^+$.

Then, for any $F: \mathbb{R}^n \rightarrow \mathbb{R}^d$ in $L^2(g) \cap C^1$ we have

$$\int \|F(x) - Z^{-1} \int g F\|_{g(x)}^2 dx$$

$$\leq \int \langle -(g(x))^{-1} \nabla F, \nabla F \rangle_{Id_n \otimes g} dx$$

Ideas from

Prior result and proof by Hörmander.

introduce

Proof sketch: $\mathcal{L} F = \Delta F = \sum_{j=1}^n (g(x)^{-1} \partial_j g) \partial_j F$

Facts ① $F, G \in C^1$ compactly supported

$$\int_{\mathbb{R}^n} \langle \mathcal{L} F, H \rangle_g = - \int \langle \nabla F, \nabla H \rangle_{Id_n \otimes g}$$

Fact 2. Bochner formula

$$\int \| \nabla F \|_g^2 = \int \langle -\Theta^\alpha \nabla F, \nabla F \rangle_{\text{Id}_n \otimes g} \\ + \int \sum_{j,k=1}^n \| \partial_{j\bar{k}} F \|_g^2$$

Proof of Prekopa-Rauzi

$V_0 \in [v_1, \dots, v_n] \in M_{n_0, d}$. At every $t_0 \in \mathbb{R}^{n_0}$

$$\langle \Theta^\alpha V_0, V_0 \rangle_\alpha \leq 0$$

||

$$\int_{\mathbb{R}^{n_0}} \langle \Theta_{00}^\alpha V_0, V_0 \rangle_{\text{Id}_n \otimes g} dy$$

$$= \int \| F^{-\alpha}(t_0)^{-1} \int g F \|_{g(t_0, y)}^2 dy$$

$$F = \sum_{j=1}^n (\Theta^{-1} d_{x_j} g) v_j \\ (g = e^{-\psi})$$

$$\nabla_y F = \Theta_{i0}^\alpha V_0$$

$$\Theta^\alpha = \left[\begin{array}{c|c} \Theta_{00}^\alpha & \Theta_{0i}^\alpha \\ \hline \Theta_{i0}^\alpha & \Theta_{ii}^\alpha \end{array} \right]$$

$$\Theta_{00}^\alpha \in L(M_{n_0, d})$$

$$\Theta_{i0}^\alpha \in L(M_{n_0, d}, M_{n_0, d})$$

$$\Theta_{ii}^\alpha \in L(M_{n_0, d})$$

Prop. at any (t, y) and for any

$$V_0 \in [v_1, \dots, v_n]$$

$$\langle (-\Theta_{00}^\alpha) V_0, V_0 \rangle_{\text{Id}_{n_0} \otimes g} \leq \langle (-\Theta_{ii}^\alpha)^{-1} \Theta_{00}^\alpha V_0, \Theta_{00}^\alpha V_0 \rangle_{\text{Id}_{n_0} \otimes g}$$

generalized Monge-Ampere equation for matrix valued weights.