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When is a group determined by its boundary?

- Quasi-moebius maps between Morse boundaries of CAT(0) spaces.

w/ Devin Murray

Rk: so a boundary is determined by a group, but we are interested in the inverse question: Given boundary what is group?

Motivating question: To what extent is a group G determined by ∂G?

Rk: many different ways to define ∂G.

case 1: G: hyperbolic group.

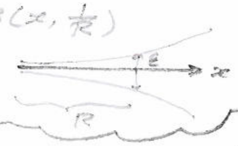
X: hyp. metric space, proper (may be droppable)

∂X = visual boundary

= $\{\alpha :$ geodesic rays$\}/\sim$

topology: rays ε-fellow travel for distance R. $\varepsilon > 2\delta$.

$B(x, \frac{1}{R})$



Gromov: if $X \xrightarrow[q.i.]{} Y$

then the induced map

$\partial X \longrightarrow \partial Y$ is a homeomorphism.

Hence, ∂G is well-defined.

Paulin (1996): X, Y both proper, cocompact hyperbolic spaces with $\partial X \xrightarrow[\text{homeo.}]{F} \partial Y$

TFAE

(1) F is induced by a q.i. $X \xrightarrow{f} Y$

(2) F is a quasi-moebius map.

((3) F is quasi-conformal)

will ignore this today: "Annuli map in an undistorted way"

will focus on this

need the following:

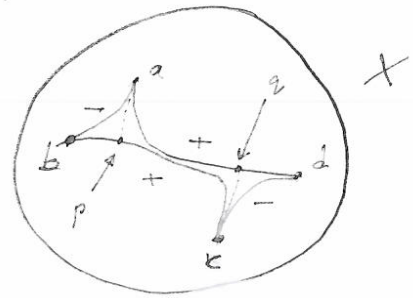
The CROSS-RATIO of $a, b, c, d \in X$ is

$$[a,b,c,d] = \frac{1}{2}\left|\left(d(a,d) + d(b,c) - d(a,b) - d(c,d)\right)\right|$$

$$\cong d(p,q)$$

in literature, abs. value is on other side.

where:



on ∂X

$$[\alpha, \beta, \gamma, \delta] = \sup\{\liminf_i [a_i, b_i, c_i, d_i]\}$$

where $x_i \longrightarrow \xi_i$

a map $\partial X \xrightarrow{F} \partial Y$ is QUASI-MOEBIUS

if $\exists\, \varphi : [0,\infty) \longrightarrow [0,\infty)$ s.t.

$\forall$ quadruples $\alpha, \beta, \gamma, \delta \in \partial X$

$$[F(\alpha), F(\beta), F(\gamma), F(\delta)] \leq \varphi([\alpha, \beta, \gamma, \delta])$$

and similarly for F^{-1}

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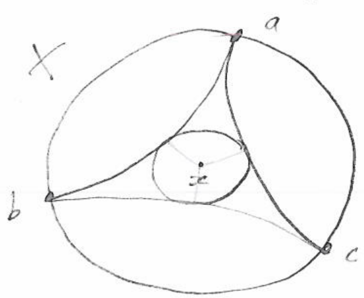
idea of proof $(2) \Rightarrow (1)$:

reminiscent of work of B. Bowditch

Construct f that induces F.

Associate to a triple in ∂X.
the barycenter of the ideal triangle spanned by the triple



$$(a,b,c) \xrightarrow{\pi_X} x.$$

$\partial X^{(3)}$ $\qquad$ X.

space of triples

$F : \partial X \longrightarrow \partial Y$ induces :

$$\begin{array}{ccc} \partial X^{(3)} & \xrightarrow{F^{(3)}} & \partial Y^{(3)} \\ \downarrow \pi_X & & \downarrow \pi_Y \\ X & \overset{f?}{\dashrightarrow} & Y \end{array}$$

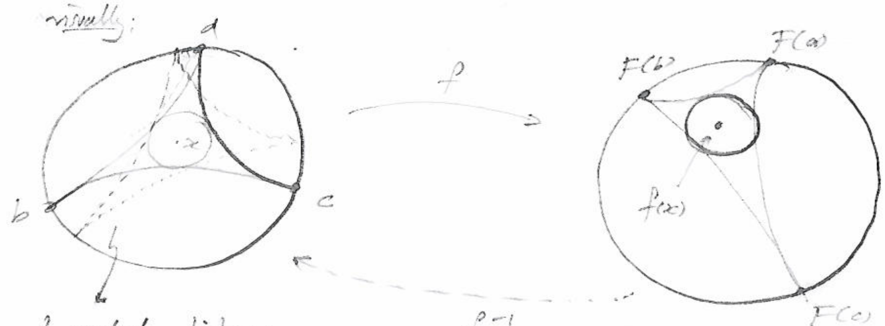
cocompact group action

Rk: more than one triangle has the same barycenter.

resolution : $\pi_Y \circ F \circ \pi_X^{-1}$ is bounded.

so f is well-defined up to quasi-isometry.

visually:



bounded distance from $\Delta(a,b,c)$

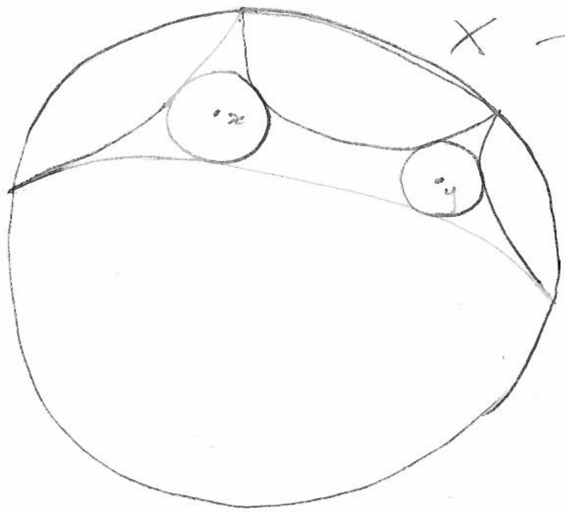
remains to see that $f : X \to Y$ is a quasi-isometry.

get this from cross ratio. and quasi-moebius.

$d(x,y) \cong [a,b,c,d]$ where

behaves well

$X \longrightarrow Y$



Rk: depends heavily on quasimoebius condition.

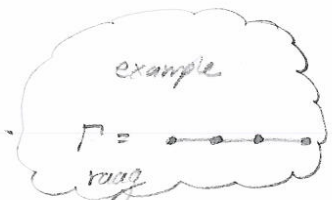
eg. if $X =$ hyperbolic space, $Y =$ complex hyperbolic space } same dimension

the above fails.

Q: Can this be generalised to CAT(0) spaces?

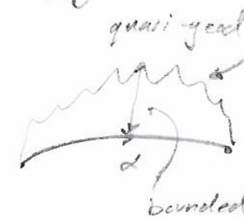
Croke + Kliener: No.

∂X is NOT q.i. invariant.



obstruction: Morse lemma fails.

quasigeodesics stay bounded distance from geodesics.



solution: redefine ∂X

the MORSE BOUNDARY is

$$\partial_* X = \{\alpha \in \partial X : \alpha \text{ is Morse}\}/\sim$$

$$= \{\text{"hyperbolic-like directions}\}$$

topology: direct limit topology

$$\partial_*^N X = \{\alpha \in \partial X : \alpha \text{ is } N\text{-Morse}\}/\sim$$

so $\partial_* X = \varinjlim_N \partial_*^N X$

Morse $\iff$ spends uniformly bounded amount of time in flats.

Rk (Murray): in general $\partial_* X$ is not metrizable

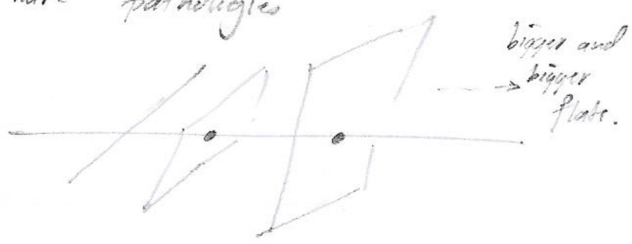
Rk: the Morse boundary may be empty.

e.g. "tree of flats." $X =$ univ. cover of

$T^2 \vee S^1$

N.B. subspace topology is insufficient.
have counterexamples (Cashen)

also have pathologies



Charney + Sultan: X, Y both proper CAT(0) spaces that are quasi-isometric.
then $\partial_* X \xrightarrow{\sim} \partial_* Y$ induced from the q.i. is a homeomorphism.

Hence, $\partial_* G$ is well-defined for G: CAT(0).

generalisation.

M. Cordes: Morse boundary can be generalised to arbitrary proper geodesic metric space

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Charney + Murray: X, Y proper, cocompact CAT(0) spaces and $\partial_* X \xrightarrow{F} \partial_* Y$ with $|\partial_* X| \geq 3$ (nontrivial) TFAE.

(1) F is induced by q.i $X \xrightarrow{f} Y$

(2) F is quasimobius and (2-stable)

↑ unnecessary? conjecturally a consequence of quasi-moebius.

Rk F is quasi-moebius if $\partial_*^N X \longrightarrow \partial_*^N Y$ is quasi-moebius $\forall N$

F is 2-stable if

$\forall N \; \exists N'$ s.t. $\partial_*^N X^{(2)} \longrightarrow \partial_*^{N'} Y^{(2)}$

double space

X