THE SHAPE OF THE MODULI SPACE
NOTES FROM THE OCTOBER 2016 MSRI WORKSHOP ON MAPPING CLASS GROUPS AND OUTER AUTOMORPHISM GROUPS

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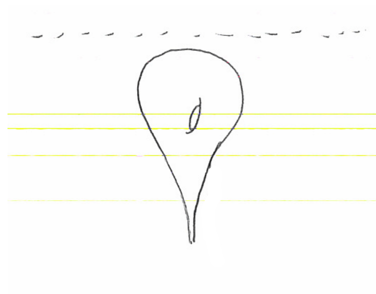
Consider $\mathcal{M}_g$ the moduli space of genus g hyperbolic surfaces. Equip it with the Teichmüller metric.

One way to understand $\mathcal{M}_g$ is with the systole function

$$syst : \mathcal{M}_g \to \mathbb{R}_+$$

With $syst(X) = \min_\alpha \text{length}_X(\alpha)$.

It is known that the systole is a topological Morse function, and its analysis gives a traditional picture of $\mathcal{M}_g$



Finite volume, infinite diameter, suggesting a unique maximum for the systole.

Today we suggest this picture is flawed.

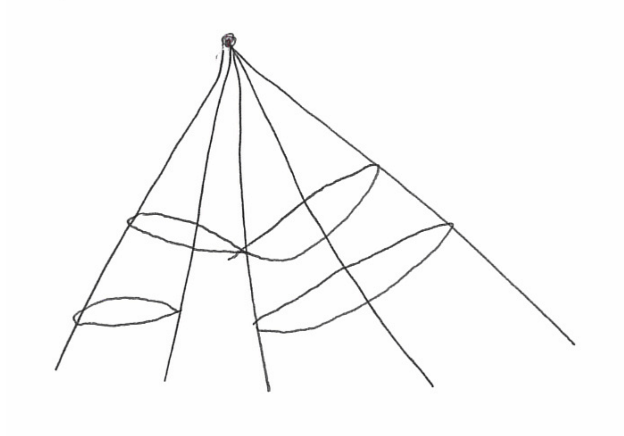
Theorem (Fourtier-Bourque–Rafi)**.** *There exists a sequence $L_n \to \infty$ so that if $g = \frac{kn}{2} + 1$ with $k > 0$ and g large ($g > a^n$) then $syst : \mathcal{M}_g \to \mathbb{R}_+$ has $N(n, g)$ local maxima at L_n where*

$$N(n, g) \geq \frac{g^g}{(cgn)^k}$$

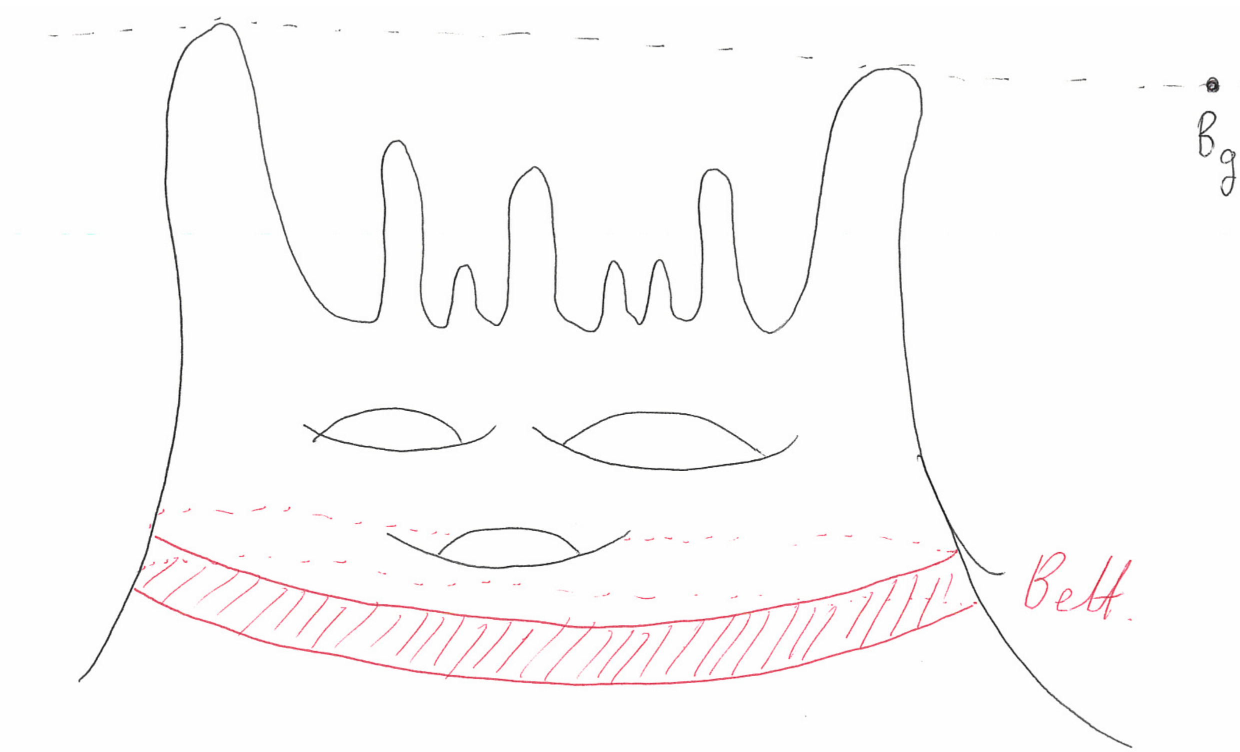
The remark is that the systole is not good for describing the topology of $\mathcal{M}_g$.

Recall Farb-Masur's result that $\mathcal{M}_g$ is $(1, D_g)$-quasi-isometric to $\frac{Cone(\mathcal{C}(S))}{Mod(S)}$, suggesting this coarse picture

Notes prepared by Edgar A. Bering IV.



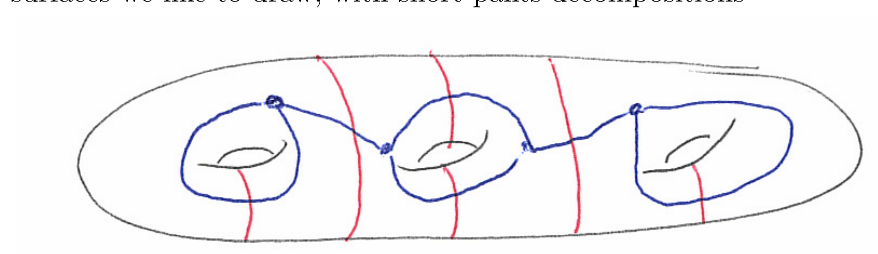
where although the picture is a fan the volume is still decreasing as $\mathcal{C}(S)$ lacks full dimensional cells. The theorem suggests that upon zooming in we arrive at this picture



Buser-Sarnak give an estimate $\frac{4}{3}\log g \leq B_g \leq 2\log g$. This

$$\text{Belt} = \{X | \exists \text{a pants decomposition } P \text{ with } length(X, \alpha) \asymp 1, \alpha \in P\}$$

These are the surfaces we like to draw, with short pants decompositions



Pants decompositions correspond to trivalent graphs. Let $G(m)$ te trivalent m vertex graphs, $|G(m)| = m^m$. This tells us the belt is combinatorially large. Specifically, Rafi-Tao show

$$diam_{Teich}(Belt) \asymp \log g$$

and so we get

$$diam_{Teich}(\mathcal{M}_g^{\geq \epsilon}) \asymp \log(g/6)$$

Continuing this line of describing $\mathcal{M}_g$ we arrive in the unknown.

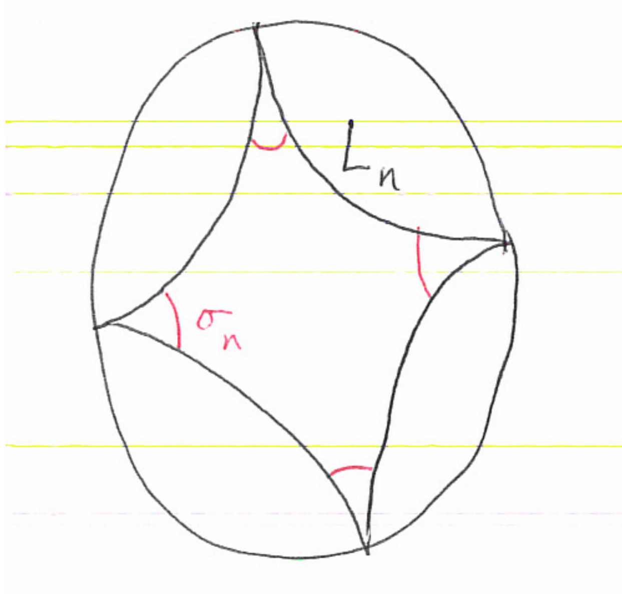
Question. (1) *Are there closed geodesic loops in the $c \log g$-thick part of $\mathcal{M}_g$*
(2) *What is the Cheeger constant of $\mathcal{M}_g$?*
(3) *What is the "depth" of $\mathcal{M}_g$, that is how far into the thin part must a geodesic joining two local maxima go?*
(4) *Is there a Collar/Margulis lemma? That is does there exist an ϵ such that for all g and $X \in \mathcal{M}_g$, $Stab_\epsilon(X)$ is elementary?*

Recall the work of Schmutz. If X is a local max then the systole curves of X have nice properties.

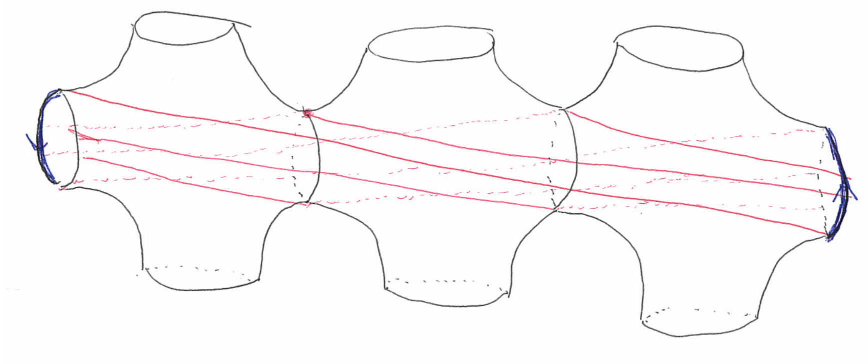
(1) $Sys(X)$ fills X.
(2) $|Sys(X)| \geq 6g - 5$. Best known examples have $12g - 12$ systoles
(3) There exists L_S for all g odd such that there is a local maximum in $\mathcal{M}_g$ at L_S.

1. Sketches of the proof $n = 3$

Basic building blocks are crosses, doubled hyperbolic squares:



with σ_n small and L_n large. When $n = 3$ consider



Glue rings of n crosses, and then get a graph of the gluing pattern. The dual is n regular. The systoles are in the diagram (and translates by symmetry).

To calculate genus, if Γ the guling graph has k vertices, then $2E_\Gamma = nk = 2(g-1)$.

So for all n there exists L_n so that for all n-regular Γ with sufficiently large girth, the systoles of X_Γ are the curves in the picture.

Theorem (Bollabás)**.** *For all j*

$$\frac{\#n\text{-regular graphs with } k \text{ vertices and girth at least } j}{\text{all of them}} \geq c_{jn}$$

So we have plenty of graphs. Looking at systoles now we can look at their shadows in Γ and the gluing graph. Shadow in Γ is trivial if $girht(\Gamma) \cdot \sigma_n > L_n$. Shadow in the gluing graph is non-trivial. These reduce the calculation and classification of systoles to a local argument.

1.1. **Why are these Maxima?** Kerckoff formula $\frac{\partial}{\partial \tau_\alpha} \ell_\beta = \cos\theta$ for intersecting curves.

There is a nice collection of reflectionally symmetric local curves where twists, along with a few lengths of "f-curves" whose derivatives are a basis for the tangent space of $\mathcal{M}_g$ that is easy to calculate in. Let F be the set of f-cruves and R the reflectionally symmetric curves. In this basis we can find a pair of inequalities that show these points are local maxima.