MONSTER GROUPS ACTING ON CAT(0) SPACES

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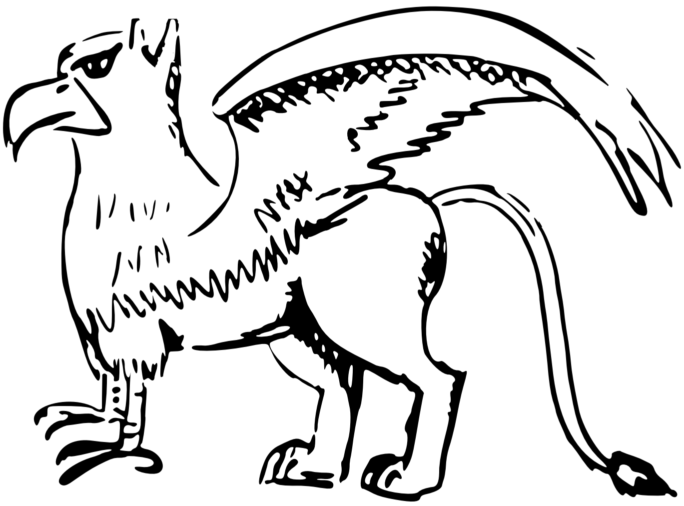
ABSTRACT. Since the beginning of the 20th century, infinite torsion groups have been the source of numerous developments in group theory: Burnside groups Tarski monsters, Grigorchuk groups, etc. From a geometric point of view, one would like to understand on which metric spaces such groups may act in a non degenerate way (e.g. without a global fixed point).

In this talk we will focus on CAT(0) spaces and present two examples with rather curious properties. The first one is a non-amenable finitely generated torsion group acting properly on a CAT(0) cube complex. The second one is a non-abelian finitely generated Tarski-like monster : every finitely generated subgroup is either finite or has finite index. In addition this group is residually finite and does not have Kazhdan property (T).

(Joint work with Vincent Guirardel)



Infinite finitely generated (head of an eagle) torsion group (body of a lion).

Throughout *torsion group* means *finitely generated infinite torsion group*.

Burnside problem: does such a group have to be finite?

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- Golod–Shafarevich 1964 gave first infinite examples.
- Free Burnside groups

  $$\mathbb{B}_r(n) = \langle a_1, \dots, a_r \mid \forall x, x^n = 1 \rangle$$

  For $r \geq 2, n \gg 1$, $\mathbb{B}_r(n)$ is infinite.

  (Novikov–Adian 1968, Ol'shanskii, . . .)
- (Ol'shanskii, 1982) Tarski monster. For all prime $p \gg 1$, there exists a finitely generated infinite group G such that every proper subgroup is $\mathbb{Z}/p$.
- (C.–Gruber) There exists a quotient of $\mathbb{B}_r(n)$ which coarsely contains expander graphs.

Question. On what space X can such a group G act in a "non-degenerate way"?

Non-degenerate:

- proper action $\forall x \in X, \forall r \geq 0, \# \{g \in G \mid d(gx, x) \leq r\} < +\infty$
- no global fixed point / bounded orbits

X tree	G fixes a point – torsion groups have property $(F\mathbb{R})$
X δ-hyperbolic	G has bounded orbit or fixes a point in ∂X
$X = \mathbb{E}^n$	G fixes a point
X proper CAT(0) cube complex	G fixes a point (Sageev)
X Hilbert	$\exists$ torsion groups with (T)

Question. Does every torsion group acting on a CAT(0) space have a global fixed point?

NO! Let G be the Grigorchuk group.

Then subexponential growth $\implies$ amenable $\implies$ a-T-menability $\implies$ $\exists$ proper action on a Hilbert space.

Question. Take Γ a CAT(0) group. May Γ contain an infinite torsion subgroup?

If we believe the Tits alternative (open conjecture whether it holds for CAT(0) groups), then the answer should be no.

Now since the free Burnside group is non-amenable, we should try that.

Theorem (Osajda, 2016)**.** *If $r \geq 2, n \gg 1$ not prime, then $\mathbb{B}_r(n)$ acts without bounded orbits on a CAT(0) cube complex.*

(Thus they do not have property (T), which was open for 20 years.)

Construction: take a clever double cover $X' \to X = \mathrm{Cay}(\mathbb{B}_r(n))$, and being a double cover means we can build walls, and thus there is a corresponding CAT(0) cube complex. Then $\mathbb{B}_r(2n) \curvearrowright X'$. This does not contradict Sageev because the cube complex is infinite dimensional.

It is unknown whether we can make the action proper, whether free Burnside groups have Haagerup property.

From now on, all examples have unbounded torsion.

Theorem (C.–Guirardel)**.** *There exists a finitely generated torsion group G such that*

(1) G is non-amenable,

(2) G acts properly on an infinite-dimensional CAT(0) cube complex,

(3) G is residually finite.

"Quasi-Tarski Monsters"

We want something like Tarski monsters, but residually finite (which requires finite index subgroups, which Tarski monsters do not have, since all their proper subgroups are finite).

Theorem (C.–Guirardel)**.** *There exists a finitely generated torsion group G such that*

(1) every f.g. subgroup of G is either finite or has finite index,

(2) G is residually finite,

(3) G does not have property (T).

Via abstract nonsense, this implies G is LERF.

SMALL CANCELLATION THEORY

Gromov, Delzant, Champetier, Ol'shanskii, ...

Prop A: G non-elementary hyperbolic group. Let $g \in G$. "For large n", $\overline{G} = G/\langle\langle g^n \rangle\rangle$ is hyperbolic non-elementary.

$\mathbb{F}_2 = \{g_0, g_1, \dots\}$, take $g_0 = 1$.

$$G_0 = \mathbb{F}_2 \twoheadrightarrow G_1 \twoheadrightarrow G_2 \twoheadrightarrow \dots \twoheadrightarrow G_\infty = \varinjlim G_k$$

where each map $G_{i-1} \twoheadrightarrow G_i$ is kill $g_i^{n_i}$.

G_∞ is by construction an infinite f.g. torsion group (cannot be finite as this would imply finitely presented, and the above sequence of quotients would stabilize).

Prop B: G non-elementary hyperbolic group, H a non-elementary subgroup of G (plus something about finite normal subgroups).

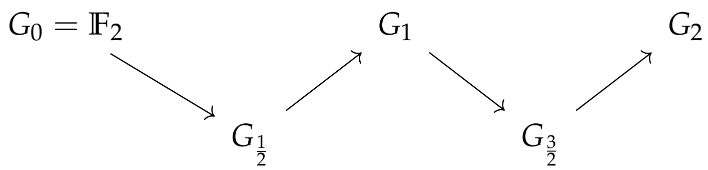
$\forall\, S \subset G$ finite, $\exists\, \overline{G}$ quotient of G such that

(1) $\overline{G}$ hyperbolic non-elementary

(2) $G \to \overline{G}$ maps H onto $\overline{G}$, one-to-one when restricted to S

$\mathbb{F}_2 = \{g_0, g_1, \dots\}$

List pairs $(u_0, v_0), (u_1, v_1), \dots$



For $G_k \to G_{k+\frac{1}{2}}$, we kill $g_{k+1}^{m_k+1}$.

For $G_{k+\frac{1}{2}} \to G_{k+1}$, consider $\langle u_{k+1}, v_{k+1} \rangle$, and ask whether it is cyclic. If yes, then do nothing. If no, then make $\{u_{k+1}, v_{k+1}\}$ generate.

Theorem (Wise's Malnormal Special Quotient Theorem, Wise, Agol—Groves–Manning)**.** *Let G be a non-elementary hyperbolic group acting geometrically on a CAT(0) cube complex X. Let H be almost malnormal, quasi-convex. There exists a finite-index subgroup $H_0 <_{\text{f.i.}} H$, such that* *for all $H_1 <_{\text{f.i.}} H_0$, $\overline{G} = G/\langle\langle H_1 \rangle\rangle$ is non-elementary hyperbolic and acts geometrically on a CAT(0) cube complex.*

We want to use $H = \langle g \rangle$.

The difficulty is trying to take a limit of the CAT(0) cube complexes on which the group $G_1, G_2, \ldots$ act, which have increasingly many elliptic elements.

C.–Guirardel proved the variation (*italics* indicating additions)

Prop A′: G non-elementary hyperbolic group *acting geometrically on X a CAT(0) cube complex.* Let $g \in G$.

"For large n", *there exists $\overline{X}$ a CAT(0) cube complex such that*

(1) $\overline{G} = G/\langle\langle g^n \rangle\rangle$ is hyperbolic non-elementary *acting on $\overline{X}$ geometrically*.

(2) $f : X \to \overline{X}$ "control line geometry"

Likewise:

Prop B′: $G = \langle \mathcal{S} \rangle$ non-elementary hyperbolic group *acting geometrically on X*, $h_1, h_2 \in H$ a non-elementary subgroup of G, *and no finite index subgroup of H stabilizes a hyperplane of X* (plus something about finite normal subgroups).

$\forall\, S \subset G$ finite, $\exists\, \overline{G}$ quotient of G *and $\exists\, \overline{X}$ a CAT(0) cube complex* such that

(1) $\overline{G}$ hyperbolic non-elementary

(2) $G \to \overline{G}$ maps H onto $\overline{G}$, one-to-one when restricted to S

(3) $f : X \to \overline{X}$, $G/\langle\langle r_1, r_2, \ldots \rangle\rangle$, $r_i = \mathcal{S}w(h_1, h_2)$