ARITHMETIC GROUPS: GEOMETRY AND COHOMOLOGY

KEVIN WORTMAN

I. Bieri–Eckmann Duality

- F a field of coefficients, henceforth implicit.
- Γ a discrete group.

Definition. $H^*(\Gamma) = H^*(K(\Gamma, 1))$.

Duality

If Γ acts *freely* and *cocompactly* on a *contractible* complex X with $H_c^n(X) = 0$ if $n \neq d$ for some d, then Γ is a *d-dimensional duality group* and for all k,

$$H_k(\Gamma, H_c^d(X)) \cong H^{d-k}(\Gamma).$$

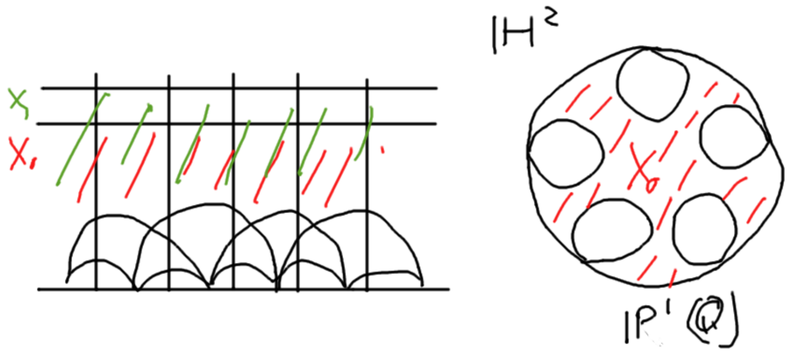
We call $H_c^d(X)$ the *dualizing module*.

II. $\mathrm{SL}_2\,\mathbb{Z}$ and Borel–Serre

- $\mathrm{SL}_2\,\mathbb{R}$ acts properly on $\mathbb{H}^2$.
- $\mathbb{Z} \leq \mathbb{R}$ is discrete.

Thus $\mathrm{SL}_2\,\mathbb{Z} \leq \mathrm{SL}_2\,\mathbb{R}$ is discrete so $\mathrm{SL}_2\,\mathbb{Z}$ acts properly on $\mathbb{H}^2$.

Date: 26 August 2016.



The action is not cocompact, but the fix is easy: take a compact subset of a fundamental domain, then take all its translates.

X_0 is the space you get after removing horoballs, one for every point in $\mathbb{P}^1(\mathbb{Q})$, from $\mathbb{H}^2$.

$\mathrm{SL}_2\,\mathbb{Z}$ acts cocompactly on X_0, and on each X_n.

$X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots \subseteq \hat{\mathbb{H}}^2$, where $\hat{\mathbb{H}}^2 \cong_{homeo} X_n$ is $\mathbb{H}^2$ augmented by a line $\mathbb{R}$ for each $x \in \mathbb{P}^1(\mathbb{Q})$.

- $\partial \hat{\mathbb{H}}^2 = \bigsqcup_{\mathbb{P}^1(\mathbb{Q})} \mathbb{R} \simeq_{homotopic} \mathbb{P}^1(\mathbb{Q})$ discrete.
- $\mathrm{SL}_2\,\mathbb{Z}$ acts cocompactly on $\hat{\mathbb{H}}^2$, the *Borel–Serre* bordification (implicitly: of $\mathbb{H}^2$ with respect to the $\mathrm{SL}_2\,\mathbb{Z}$ action).
- $\mathrm{SL}_q\,\mathbb{Q}$ acts on $\hat{\mathbb{H}}^2$.

Duality. Finite-index torsion-free (fitf) $\Gamma \leq \mathrm{SL}_2\,\mathbb{Z}$ act freely on $\widehat{\mathbb{H}}^2$, so Γ is a 1-dimensional duality group:

$$H^n_c(\widehat{\mathbb{H}}^2) = H_{2-n}(\widehat{\mathbb{H}}^2, \partial\widehat{\mathbb{H}}^2) = \widetilde{H}_{1-n}(\mathbb{P}^1(\mathbb{Q}))$$

The first equality is from Lefschetz, the second from a long exact sequence pair.

III. $\mathrm{SL}_q\,\mathbb{Z}[1/p]$ AND EUCLIDEAN BUILDINGS

$\mathbb{Q}_p$ is a discretely valued field.

Complete with respect to the norm $|\frac{n}{m}p^k|_{\mathbb{Q}_p} = p^{-k}$ (p does not divide n, m).

$\mathbb{Q}_p^\times \twoheadrightarrow \mathbb{Z}$

$\mathbb{Z}[1/p] \overset{\Delta}{\hookrightarrow} \mathbb{R} \times \mathbb{Q}_p$ is a discrete embedding.

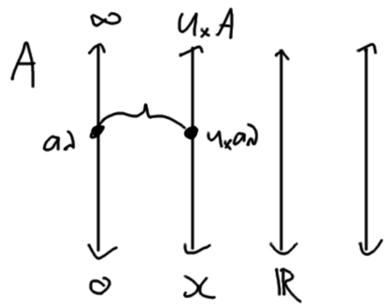
$\implies$ we have a discrete embedding

$$\mathrm{SL}_2 \mathbb{Z}[1/p] \overset{\Delta}{\hookrightarrow} \mathrm{SL}_2 \mathbb{R} \times \mathrm{SL}_2 \mathbb{Q}_p.$$

$\implies$ $\mathrm{SL}_2 \mathbb{Z}[1/p]$ acts properly on $\mathbb{H}^2 \times T_p$, a $(p+1)$-regular tree.

Let $u_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$, $a_\lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$.

$\mathbb{H}^2$: Let $A = \{a_\lambda \mid \lambda \in (0, \infty)\}$.

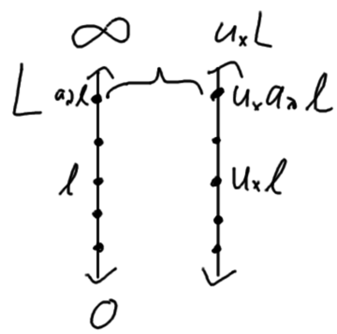


$d(a_\lambda, u_x a_\lambda) = d(1, a_\lambda^{-1} u_x a_\lambda) = d(1, u_{x/\lambda^2}) \to 0$ as $|\lambda|_{\mathbb{R}} \to \infty$.

$\mathrm{SL}_2 \mathbb{R}$ acts on $\bigcup_{x \in \mathbb{R}} u_x A = \mathbb{H}^2$.

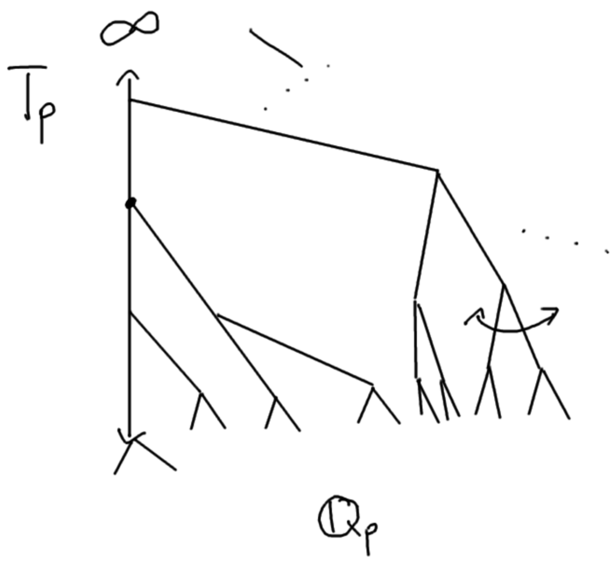
T_p: Let $A = \left\{ a_\lambda \,\middle|\, \lambda \in \mathbb{Q}_p^\times \right\}$.

Let $L = \mathbb{R}$



$A \twoheadrightarrow \mathbb{Z}$ acts by integer translations on L.

$d(a_\lambda l, u_x a_\lambda l) = d(l, a_\lambda^{-1} u_x a_\lambda l) \to 0$ as $|\lambda|_{\mathbb{Q}_p} \to \infty$.



fitf $\Gamma \leq SL_2 \mathbb{Z}[1/p]$ act freely, cocompactly, on $\widehat{\mathbb{H}}^2 \times T_p$, so Γ is a 2-dimensional duality group:

$$H_c^*(\widehat{\mathbb{H}}^2 \times T_p) = H_c^*(\widehat{\mathbb{H}}^2) \otimes H_c^*(T_p).$$

Borel–Serre (1974, 1976): fitf subgroups of arithmetic groups (e.g. $SL_n \mathbb{Z}$, $SL_n \mathbb{Z}[\sqrt{2}]$) and S-arithmetic groups (e.g. $SL_n \mathbb{Z}[1/p]$) are duality groups.

$\sim$ break $\sim$

IV. $SL_2(\mathbb{F}_p[t])$, SEMIDUALITY

(joint with Studenmund)

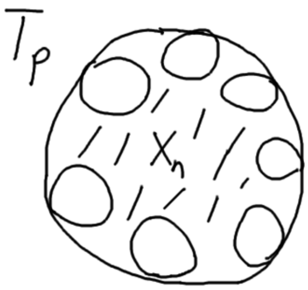
Below, $\mathrm{char}(F) \neq p$.

$\mathbb{F}_p[t] \leq \mathbb{F}_p((t^{-1}))$ discrete:

$\implies SL_2 \mathbb{F}_p[t] \leq SL_2 \mathbb{F}_p((t^{-1}))$ discrete

$\implies SL_2 \mathbb{F}_p[t]$ acts properly on T_p.

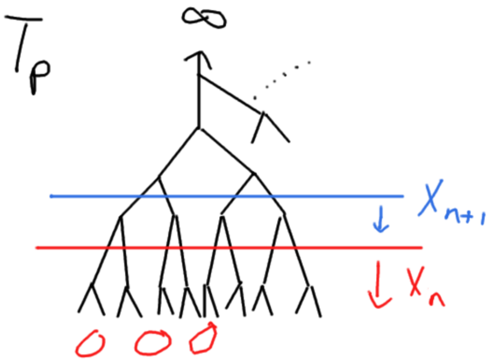
Not cocompactly, but acts freely enough for finite-index non-p-torsion-free $\Gamma \leq \mathrm{SL}_2\,\mathbb{F}_p[t]$ (only torsion is p-torsion).



$X_0 \subseteq X_1 \subseteq X_2$

$\mathrm{SL}_2\,\mathbb{F}_p[t]$ acts cocompactly on each X_n, but X_n is not contractible.

$\mathbb{F}_p((t^{-1}))$



$\partial X_n \to \partial X_{n+1}$

$H^0_c(\partial X_{n+1}) \hookrightarrow H^0_c(\partial X_n) \overset{\delta}{\hookrightarrow} H^1_c(T_p)$

$\bigcap_{n=0}^{\infty} H^0(\partial X_n) = 0.$

Let $\widehat{H^1_c(T_p)} = \varprojlim H^1_c(T_p)/H^0_c(\partial X_n)$

$$\widehat{H^1_c(T_p)} \cong H^1_c(T_p) \oplus \left(\bigoplus_{x \in \mathbb{P}^1(\mathbb{F}_p(t))} V_x \right)$$

$\mathrm{SL}_2(\mathbb{F}_p(t))$ acts on $\widehat{H^1_c(T_p)}$.

Conjecture.

- X Euclidean building of dimension d
- $G(\mathcal{O}_S)$ arithmetic group over function fields.
- $G(\mathcal{O}_S)$ acts on X as a lattice.
- K be the fraction field of $\mathcal{O}_S$.

then there is $H_n(\Gamma, \widehat{H^d_c(X)}) \to H^{d-n}(\Gamma)$ isomorphism if $n \neq d, d-1$, surjection if $n = d-1$, and $G(K)$ acts on $\widehat{H^d_c(X)}$ where $\Gamma \leq G(\mathcal{O}_S)$ is finite-index and non-p-torsion-free.

Theorem (Studenmund–W.). *Conjecture is true if* $G = \mathrm{SL}_2$.

(Actually works for any arithmetic group that acts on a product of trees.)

Audience question: what is V_x? $V_x \cong \bigoplus_{\mathbb{R}} F$.

Audience question: have you been able to do any group cohomology calculations using the dualizing module? Not yet.

Audience question on finiteness properties in the isomorphism of the conjecture.