RECOGNIZING 3-MANIFOLD GROUPS BY THEIR FINITE QUOTIENTS

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ABSTRACT. This talk will be focused on the problem of: to what extent can the fundamental groups of compact 3-manifolds be distinguished by the finite quotients of their fundamental groups. The talk will highlight examples (e.g. the figure eight knot complement) and introduce ideas and techniques used in attacking the problem.

This talk features applications of the work of Agol and Wise to very concrete problems about the finite quotients of 3-manifold groups.

All groups will be finitely generated (usually finitely presented) and residually finite (RF).

Definition (RF). $\forall 1 \neq g \in G, \exists \varphi : G \to A, |A| < \infty$ such that $\varphi(g) \neq 1$.

Example.

(1) f.g. subgroups of $GL_n(\mathbb{Z})$

(2) π_1(compact 3-manifold)

Definition. $\mathcal{C}(G) = \{A : |A| < \infty, G \twoheadrightarrow A\}$ ("list of finite quotients")

Question. To what extent does $\mathcal{C}(G)$ determine G?

Question. If $\mathcal{C}(G) = \mathcal{C}(H)$, what similarities and differences might G and H have?

Interest in $\mathcal{C}(G)$:

- group theorist: proving / deciding group is non-trivial.

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- topologist: enumerating finite covers to check whether manifolds are non-homeomorphic.

Notation. *Genus* of Γ. $\mathcal{G}(\Gamma) = \{H : \mathcal{C}(H) = \mathcal{C}(\Gamma)\}$

The use of 'genus' comes from integral quadratic forms: $q_1 q_2/\mathbb{Z}$ – to be in same genus the forms are equivalent "locally" for all p and over $\mathbb{R}$.

What might this mean here :-

Fix $\Gamma = \langle S \rangle, \quad H = \langle S' \rangle$.

Local data = finite quotient Cayley graph

Question. How much of this local data picks out global structure of Γ?

Question.

For which Γ is $\mathcal{G}(\Gamma) = \{\Gamma\}$?

For which Γ is $|\mathcal{G}(\Gamma)| > 1$?

How big can $|\mathcal{G}(\Gamma)|$ be?

Example. $\mathcal{G}(\mathbb{Z}) = \{\mathbb{Z}\}$.

Why? Let $\Delta \in \mathcal{G}(\mathbb{Z})$. Assume Δ is non-abelian. Can find $c = [a, b] \neq 1 \in \Delta$. Δ is RF $\implies \exists\, \varphi : \Delta \to A$, $|A| < \infty$, with $\varphi(1) \neq 1$. Contradiction – all finite quotients are cyclic.

Structure Theorem for f.g. abelian groups $\implies \Delta \cong \mathbb{Z}$.

Note.

(1) This argument also shows $\mathcal{G}(\Gamma) = \{\Gamma\}$ for Γ any f.g. abelian group.

(2) You can see the rank of f.g. abelian group Γ in finite quotients.

(3) Γ, Δ with $\mathcal{C}(\Gamma) = \mathcal{C}(\Delta)$ then $\Gamma^{ab} \cong \Delta^{ab}$.

Constructions of $\Gamma \ncong \Delta$ with $\mathcal{C}(\Gamma) = \mathcal{C}(\Delta)$.

Stability. (Baumslag, early 1970's)

(Baumslag's student Pickel completely understood the nilpotent case.)

Stability Lemma (Baumslag). Let G, H be f.g. groups. If $G \times \mathbb{Z} \cong H \times \mathbb{Z} \implies \mathcal{C}(G) = \mathcal{C}(H)$.

Stability of Extensions. Let N be a f.g. group and $G_\varphi = N \rtimes_\varphi \mathbb{Z}$ where φ is a periodic outer automorphism of N of order n. Then if $(k, n) = 1$ then $G_\varphi \times \mathbb{Z} \cong G_{\varphi^k} \times \mathbb{Z}$.

Baumslag. Take $m = 11$, note $\mathbb{Z}/m\mathbb{Z}$ has an automorphism of order $\neq 1, 2, 3, 4, 5, 6$.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z}/m\mathbb{Z} & \longrightarrow & \Gamma_1 & \longrightarrow & \mathbb{Z} & \longrightarrow & 1 \\ & & & & \not\cong & & & & \\ 1 & \longrightarrow & \mathbb{Z}/m\mathbb{Z} & \longrightarrow & \Gamma_2 & \longrightarrow & \mathbb{Z} & \longrightarrow & 1 \end{array}$$

Baumslag inputs such that $\Gamma_1 \ncong \Gamma_2$,

Hempel. F closed orientable surface, $\varphi : F \to F$ periodic automorphism of F. The mapping torus M_φ with group G_φ can be constructed so that $G_\varphi \ncong G_{\varphi^k}$. These are Seifert fiber spaces (SFS) with the same finite quotients.

Remark.

(1) G. Wilkes (student of M. Lackenby). If M is a closed orientable SFS and N is a compact 3-manifold with $\mathcal{C}(\pi_1 M) = \mathcal{C}(\pi_1 N)$. Then either

- $N \cong M$, or
- N is an $\mathbb{H}^2 \times \mathbb{R}$ manifold and M and N are as above.

(2) Turaev, Kwasik–Rosicki

M, N are geometric 3-manifolds with $M \times S^1 \cong N \times S^1$. Then $M \cong N$ unless M, N as above.

(3) There are other closed 3-manifold that are not determined by their finite quotients. This comes from Number Theory.

(Stebe, Funar) T^2-bundles over S^1 with infinite order monodromy.

$\mathrm{SL}_2(\mathbb{Z}) = \mathrm{MCG}(T^2)$

$\exists\, \theta_1, \theta_2 \in \mathrm{SL}_2(\mathbb{Z})$ hyperbolic with $M_{\theta_1} \ncong M_{\theta_1}$ and $\mathcal{C}(\pi_1 M_{\theta_1}) = \mathcal{C}(\pi_1 M_{\theta_2})$.

$$\theta_1 = \begin{pmatrix} 188 & 275 \\ 121 & 177 \end{pmatrix}, \theta_2 = \begin{pmatrix} 188 & 11 \\ 3025 & 177 \end{pmatrix}$$

Number theory comes in to show that θ_1 and θ_2 are conjugate in $\mathrm{SL}_2(\mathbb{Z}/m\mathbb{Z})\, \forall\, m$ but not in $\mathrm{SL}_2(\mathbb{Z})$.

Number theory can be used to construct lattices in higher rank semi-simple Lie groups with $|\mathcal{G}(\Gamma)| > 1$.

Stability. $G \times \mathbb{Z} \cong H \times \mathbb{Z} \implies \mathcal{C}(G) = \mathcal{C}(H)$.

This uses the Remak–Krull–Schmidt Theorem.

Notation. G a f.g. group. $G(n) = \cap\{N \lhd G : [G : N] \le n\}$. To prove $\mathcal{C}(G) = \mathcal{C}(H)$, it suffices to show that

$$G/G(n) \cong H/H(n) \quad \forall\, n$$

Check. $(G \times \mathbb{Z})(n) = G(n) \times \mathbb{Z}(n)$

$$\begin{array}{ccc} (G \times \mathbb{Z})/G(n) \times \mathbb{Z}(n) & = & G/G(n) \times \mathbb{Z}/\mathbb{Z}(n) \\ \text{\rotatebox{90}{$\cong$}} & & \\ (H \times \mathbb{Z})/H(n) \times \mathbb{Z}(n) & = & H/H(n) \times \mathbb{Z}/\mathbb{Z}(n) \end{array}$$

Then Remak–Krull–Schmidt Theorem allows one to deduce $G/G(n) \cong H/H(n)$.

Focus on low-dimensional examples.

- free groups
- surface groups
- π_1(finite volume hyperbolic 3-manifolds)

Open.

- $\mathcal{G}(F_n) = \{F_n\}, n \geq 2$? (Remeslennikov 1971)
- $\mathcal{G}(\pi_1\Sigma_g) = \{\pi_1\Sigma_g\}$? $g \geq 2$
- $\mathcal{G}(\pi_1(M \text{ a finite volume hyperbolic 3-manifold})) = \{\pi_1 M\}$?

Talk about progress on these questions. *This uses Agol, Wise.*

Example. Distinguish free group from a surface group. We need to distinguish F_{2g} from $\pi_1\Sigma_g$ (using the abelianization).

For $F_{2g} \xrightarrow{\varphi} \mathbb{Z}/2\mathbb{Z}$, the kernel K is a free group of rank $4g-1$.

Now map $K \xrightarrow{\psi} (\mathbb{Z}/p\mathbb{Z})^{4g-1}$, p prime.

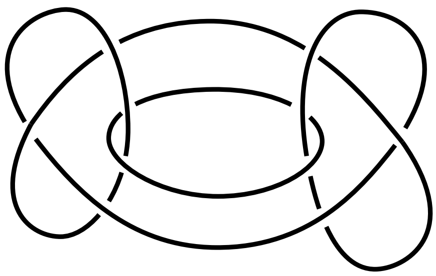
Let $H = \ker\psi$, which is characteristic hence normal.

$F_{2g}/H =$ a finite group with $(\mathbb{Z}/p\mathbb{Z})^{4g-1}$ as index 2 subgroup.

But this cannot be surjected by $\pi_1\Sigma_g$.

Distinguish F_2 from finite volume hyperbolic 3-manifolds.

Example that's tricky: link L in S^3, $S^3 \setminus L = \mathbb{H}^3/\Gamma$

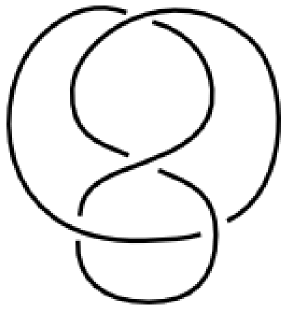


This has all 2-generator finite groups as quotients.

Worked example.

The figure eight knot.

FIGURE 1: Figure eight knot, source: Makotoy https://commons.wikimedia.org/wiki/File:Fig8_knot_rp.png



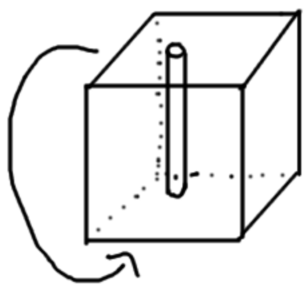
$\Gamma = \pi_1(S^3 \setminus K)$. $S^3 \setminus K = \mathbb{H}^3/\Gamma$, $\Gamma = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \omega & 1 \end{pmatrix} \right\rangle$ (has entries in $\mathbb{Z}[\omega]$) where $\omega^2 + \omega + 1 = 0$.

Theorem (Bridson–R., Boileau–Friedl). *Let N be a compact 3-manifold with $\mathcal{C}(\pi_1 N) = \mathcal{C}(\Gamma)$. Then $N \cong S^3 \setminus K$.*

Write $\Delta = \pi_1 N$.

Preliminary Comments.

- $\Gamma^{ab} \cong \mathbb{Z} \implies \Delta^{ab} \cong \mathbb{Z}$.
- Γ has lots of non-abelian finite quotients. For example, reduction module primes in $\mathbb{Z}[\omega]$, surjects onto lots of $\mathrm{SL}_2(\mathbb{F}_p)$.
- $S^3 \setminus K$ has the structure of a once-punctured T^2-bundle over S^1



$$1 \to \langle a, b \rangle \to \Gamma \to \mathbb{Z} \to 1$$

$$\theta = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

- N is orientable.
- N can't be a closed 3-manifold *uses Agol, Wise*.
- N can't be a connect sum (irreducible) *uses L^2-methods*.
- $\partial N \neq \emptyset \implies \partial N = T^2$ "half lives, half dies", T^2 incompressible

Reduced N to compact orientable irreducible $\partial N = T^2$ incompressible.

Goal. Show that N is fibered $\to$ it has to be fibered with fiber a once-punctured T^2. Then easy to finish: only such examples with $H_1 = \mathbb{Z}$ are the figure eight and trefoil knot complements (and Gieseking manifold, if we allow non-orientable). Example: $\Gamma \to D_5$ but $\pi_1(\text{trefoil}) \not\to D_5$.

Organizing Finite Quotients – Profinite Completion

Write $\mathcal{N} = \{\text{finite subgroups of finite index}\}$.

$$\begin{aligned}\hat{\Gamma} &= \varprojlim_{N \in \mathcal{N}} \Gamma/N \\ &= \{(\gamma_N) \in \prod \Gamma/N : \forall\, M, N \text{ where } M \subset N, f_{MN}(\gamma_M) = \gamma_N\}\end{aligned}$$

Theorem. Γ_1, Γ_2 *f.g., then* $\mathcal{C}(\Gamma_1) = \mathcal{C}(\Gamma_2) \iff \hat{\Gamma}_1 \cong \hat{\Gamma}_2$. *(This isomorphism is now just an abstract isomorphism, not a topological isomorphism, by work of Nikolov–Segal.)*

Original Questions. When does $\hat{\Gamma}_1 \cong \hat{\Gamma}_2 \implies \Gamma_1 \cong \Gamma_2$?

Remark. Grothendieck's Problem (Bridson–Grunewald).

$\iota : P \hookrightarrow \Gamma$ induces isomorphism $\hat{P} \cong \hat{\Gamma}$, but $P \ncong \Gamma$.

Question. Does there exist Γ hyperbolic with Δ hyperbolic and $\Delta \in \mathcal{G}(\Gamma)$?

Assume N not fibered. $N = \mathbb{H}^3/\Delta$ finite volume.

$H \cong \pi_1\Sigma_g \subset \ker = K \subset \Delta \to \mathbb{Z}$ with K corresponding to cover $\widetilde{N} \to N$.

There is π_1-injective embedded closed surface in $\widetilde{N}$.

Freedmen (M. + B.)

(Wise): H and all its finite index subgroups are separable in Δ

$$1 \to \overline{K} \to \hat{\Delta} \to \hat{\mathbb{Z}} \to 1$$

and $\overline{H} \subset \overline{K}$, $\overline{H} = \hat{H}$ (uses LERF).

$$\begin{array}{ccccccccc}
 & & \langle a,b\rangle & & & & & & \\
 & & \parallel & & & & & & \\
1 & \longrightarrow & F & \longrightarrow & \Gamma & \longrightarrow & \mathbb{Z} & \longrightarrow & 1 \\
 & & & & \big\downarrow & & & & \\
1 & \longrightarrow & \hat{F} & \longrightarrow & \hat{\Gamma} & \longrightarrow & \hat{\mathbb{Z}} & \longrightarrow & 1 \\
 & & \parallel & & \wr\parallel & & & & \\
1 & \longrightarrow & \overline{K} & \longrightarrow & \hat{\Delta} & \longrightarrow & \hat{\mathbb{Z}} & \longrightarrow & 1 \\
 & & \cup & & & & & & \\
 & & \hat{H} & & & & & &
\end{array}$$

Cohomological methods tell us that having such $\hat{H}$ is illegal.

Cohomological methods used to rule out closed with boundary having same profinite completions, Agol + Wise (uses 'good' in the sense of Serre).

Question. Prove that if Γ_1, Γ_2 are finite covolume Kleinian groups with $\hat{\Gamma}_1 \cong \hat{\Gamma}_2 \implies \Gamma_1 \cong \Gamma_2$.

Question. Does there exist a residually finite word hyperbolic group that is *not* good?

Audience question: what is the definition of good?

Γ is *good* if $\forall\, q, \forall$ finite module M,

$$H^q(\hat{\Gamma}, M) \xrightarrow{\cong} H^q(\Gamma, M).$$