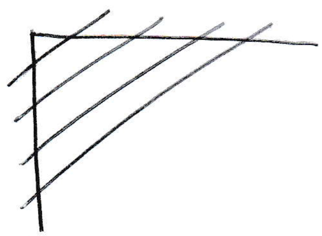
Gérard, Supplementary Boardwork

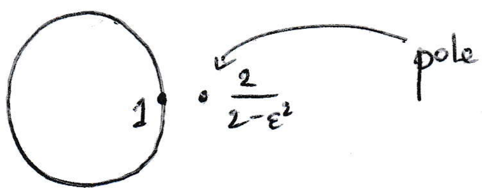
- $h \in \Pi(L^2)$, $h(z) = \sum_{k=0}^{\infty} h_k z^k$, $\sum_{k=0}^{\infty} |h_k|^2 < \infty$, $|z| \leq 1$.

- $\widehat{H_u(h)}(n) = \sum_{p=0}^{n} \hat{u}(n+p) \overline{\hat{h}(p)}$



- $\sum_{n=0}^{\infty} |\hat{u}(n)| \leq \mathrm{Tr}\sqrt{H_u^2} + \mathrm{Tr}\sqrt{K_u^2}$

$$\hat{u}(2n) = (H_u(e^{inx}) | e^{inx}), \quad \hat{u}(2n+1) = (K_u(e^{inx}) | e^{inx})$$

- 

Wave Turbulence for the cubic Szegő equation and beyond

Patrick Gérard

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New challenges in PDEs: Deterministic dynamics and randomness in high and infinite dimensional systems, Berkeley, October 19, 2015

The problem

Let

$$\frac{\partial u}{\partial t} = X(u)$$

be an infinite dimensional Hamiltonian system posed on spaces of functions on a Riemannian manifold (say the torus).

Assume the dynamics to be globally well defined on the Sobolev space H^s for s big enough (e.g. defocusing subcritical NLS, wave equation...)

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Problem : describe long time dynamics.

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In particular : do small characteristic scales appear as $t \to \infty$?

Turbulent solutions

Definition

A solution $u \in C(\mathbb{R}, H^s)$ of

$$\frac{\partial u}{\partial t} = X(u)$$

is said to be turbulent if, for some s,

$$\limsup_{t \to \infty} \|u(t,.)\|_{H^s} = +\infty \ .$$

A trivial example

Consider

$$H(u) := \frac{1}{4} \int_{\mathbb{T}} |u(x)|^4 \, dx \ ,$$

so that the Hamiltonian system reads

$$i\dot{u} = |u|^2 u \ , \ \ u(0,x) = u_0(x) \ .$$

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If $|u_0|^2$ is not a constant function,

$$\|u(t)\|_{H^s} \simeq |t|^s \ , \ \ |t| \to \infty \ .$$

A muchless trivial example

Cubic NLS, $d = 1$, $H(u) = \int_{\mathbb{T}} \left(\frac{1}{2} |\partial_x u(x)|^2 + \frac{1}{4} |u(x)|^4 \right) \, dx$,

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Zakharov-Shabat (1974) : this equation admits a Lax pair.
$\forall p \in \mathbb{N}, \exists F_p = F_p(u, \overline{u}, \dots, u^{(p-1)}, \overline{u^{(p-1)}})$ polynomial s. t.

$$\int_{\mathbb{T}} \left[|u^{(p)}(x)|^2 + F_p(u(x), \overline{u}(x), \dots, u^{(p-1)}(x), \overline{u^{(p-1)}}(x))\right]\, dx$$

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is a conservation law. No turbulent solution !

A family of 1D models

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Here we shall focus on the limit case $\alpha = 1 \, , \; \beta = 0$.

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or as a system of coupled transport equations,

$$\begin{cases} i(\partial_t u_+ + \partial_x u_+) = \Pi_+[|u_+ + u_-|^2(u_+ + u_-)] \ , \ \Pi_+ := \mathbf{1}_{D\geq 0} \ , \\ i(\partial_t u_- - \partial_x u_-) = \Pi_-[|u_+ + u_-|^2(u_+ + u_-)] \ , \ \Pi_- := \mathbf{1}_{D<0} \ . \end{cases}$$

The resonance analysis

System in Fourier coefficients

$$i\dot{u}_k = |k|u_k + \sum_{k_1-k_2+k_3=k} u_{k_1}\overline{u}_{k_2}u_{k_3}\ ,\ \ k\in\mathbb{Z}\ ,$$

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Main trend : only keep resonant quartets :

$$k_1 - k_2 + k_3 - k_4 = 0 \ , \ \ |k_1| - |k_2| + |k_3| - |k_4| = 0$$

The degeneracy of resonant quartets

(k_1, k_2, k_3, k_4) is a resonant quartet if and only if

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Resonant system is decoupled :

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Phase space : range of Π intersected with $H^{1/2}(\mathbb{T})$
$= \{$ holomorphic functions $u = u(z)$ on the unit disc $\mathbb{D}$:

$$\int_{\mathbb{D}} |u'(z)|^2 \, dL(z) < +\infty \}$$

The main theorem

Theorem (PG, S.Grellier, 2010-2015)

For every $u_0 \in \Pi\left(C^\infty(\mathbb{T},\mathbb{C})\right)$,

$$\forall t \in \mathbb{R}\ ,\ \|u(t)\|_{L^\infty(\mathbb{T})} \leq C(u_0)\ ,\ \forall s\ ,\ \|u(t)\|_{H^s} \leq C_s(u_0)\mathrm{e}^{C_s(u_0)|t|}\ .$$

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$$\forall s\ ,\ \liminf_{t\to\infty} \|u(t)\|_{H^s} < +\infty\ .$$

$$\forall s > \frac{1}{2}, \forall M \ge 1\ ,\ \limsup_{t\to\infty} \frac{\|u(t)\|_{H^s}}{|t|^M} = +\infty\ .$$

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$$\begin{aligned} \frac{d}{dt} H_u &= [B_u, H_u] \\ B_u(h) &:= -i\Pi(|u|^2 h) + \frac{i}{2} H_u^2(h) \, . \end{aligned}$$

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The eigenvalues of the trace class operators H_u^2 and $K_u^2 := H_u^2 - (.|u)u$ are conservation laws.
(provides the L^∞ estimate).

Special quasiperiodic solutions

Theorem (PG, S.Grellier, 2015)

As $d \geq 1$, $s_1 > s_2 > \cdots > s_d > 0$, $(\psi_1, \psi_2, \ldots, \psi_d) \in \mathbb{T}^d$, the following defines a dense set of solutions in $\Pi(C^\infty)$,

$$u(t,z) := \left\langle \mathscr{C}(t,z)^{-1} \begin{pmatrix} 1 \\ . \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ . \\ 1 \end{pmatrix} \right\rangle_{\mathbb{C}^N \times \mathbb{C}^N},$$

$$\mathscr{C}(t,z)_{jk} := \frac{s_{2j-1} \mathrm{e}^{i(\psi_{2j-1} + t s_{2j-1}^2)} - s_{2k} \mathrm{e}^{i(\psi_{2k} + t s_{2k}^2)} z}{s_{2j-1}^2 - s_{2k}^2},$$

with $N := \left[\frac{d+1}{2}\right]$, $s_{2N} := 0$ if $d = 2N - 1$.

$$\left\langle \begin{pmatrix} \frac{1+\varepsilon-z}{(1+\varepsilon)^2-1} & \frac{1}{1+\varepsilon} \\ \frac{-(1-\varepsilon)-z}{(1-\varepsilon)^2-1} & \frac{-1}{1-\varepsilon} \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\rangle_{\mathbb{C}^2 \times \mathbb{C}^2} = \frac{2z(1-\varepsilon^2) - 3\varepsilon}{2 - \varepsilon z}$$

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$1 \to 2$ within time interval of length

$$t = \frac{\pi}{(1+\varepsilon)^2 - (1-\varepsilon)^2} = \frac{\pi}{4\varepsilon}$$

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Instability H^s , $s > \frac{1}{2}$. Hani (2013) for resonant NLS on $\mathbb{T}^2$.

Exporting turbulent solutions to other equations

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From Szegő to the half–wave

Theorem (S. Grellier-PG, 2012 ; O. Pocovnicu, 2013)

Let $s > 1$. *For every* $\alpha > 0$, *there exists* $c_{\alpha,s} > 0$ *such that, if*

$$\Pi u_0 = u_0 = O(\varepsilon) \ \text{ in } H^s ,$$

the solutions of

$$\begin{aligned} i\partial_t u &= |D|u + |u|^2 u \ , \ i(\partial_t v + \partial_x v) = \Pi(|v|^2 v) \ , \\ u(0) &= v(0) = u_0 \end{aligned}$$

satisfy $\forall t \leq c_{\alpha,s}\varepsilon^{-2}|\log \varepsilon| \ , \ u(t) = v(t) + O(\varepsilon^{3-\alpha}) \ \text{ in } H^s \ .$

Consequence : norm inflation

Combining this approximation result with the existence of turbulent solutions and a scaling argument (O.Pocovnicu),

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Corollary

For every $\delta > 0$, for every $K > 0$ there exists a solution u to the half–wave equation and $T > 0$ such that

$$\|u(0)\|_{H^1} \leq \delta \ , \ \|u(T)\|_{H^1} \geq K \ .$$

Similar to Colliander–Keel–Staffilani–Takaoka–Tao (2010) for cubic NLS on $\mathbb{T}^2$.

Schrödinger/half–wave on the cylinder

Following Hani–Pausader–Tzvetkov–Visciglia (2013), consider the equation

$$i\partial_t u = -\partial_y^2 u + |D_x| u + |u|^2 u \ , \ (x, y) \in \mathbb{T} \times \mathbb{R} \ .$$

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Idea : the dispersion in the variable y helps in discarding the non resonant terms in the variable x.
For $s \geq 20$, introduce the following norms,

$$\begin{aligned} \|u\|_S &:= \|u\|_{H^s} + \|y\,u\|_{L^2} \\ \|u\|_{S^+} &:= \|u\|_S + \|(1-\partial_y^2)^4 u\|_S + \|y\,u\|_S \end{aligned}$$

A modified scattering result

Theorem (Haiyan Xu, 2015)

Assume that $v_0(x+\pi, y) = -v_0(x,y)$ and $\|v_0\|_{S^+} \le \varepsilon$ small enough. Consider the solution $v = v(t,x,y)$ of the system

$$
\begin{aligned}
i\partial_t \hat{v}_+(t,x,\eta) &= \Pi_+(|\hat{v}_+|^2 v_+)\ ,\ \hat{v}_+(0,x,\eta) = \Pi_+(\hat{v}_0(.,\eta))(x).\\
i\partial_t \hat{v}_-(t,x,\eta) &= \Pi_-(|\hat{v}_-|^2 v_-)\ ,\ \hat{v}_-(0,x,\eta) = \Pi_-(\hat{v}_0(.,\eta))(x).
\end{aligned}
$$

Then there exists a unique solution u of the Schrödinger/half–wave equation such that

$$
\|\mathrm{e}^{it(|D_x|-\partial_y^2)}u(t) - v(\pi \log t)\|_S \underset{t\to+\infty}{\longrightarrow} 0\ .
$$

Turbulent solutions of the Schrödinger/half-wave equation

Corollary (H. Xu, 2015)

For every s, there exist solutions of

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$$\liminf_{t \to +\infty} \|u(t)\|_{H^s} < +\infty.$$

The cubic Szegő equation on the line

Denote again by Π the operator $\mathbf{1}_{D\geq 0}$ on $L^2(\mathbb{R})$. The equation

$$i\partial_t u = \Pi(|u|^2 u)$$

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$$\forall s > \frac{1}{2}\ , \ |u(t)\|_{H^s} \simeq t^{2s-1}\ , \ t \to \infty\ .$$

Soliton interaction for Szegő on the line

Theorem (O. Pocovnicu, 2011)

The fonction

$$Q(x) := \frac{1}{x + \frac{i}{2}} \ .$$

is, up to symmetries, the only non trivial solution of

$$-i\partial_x Q + Q = \Pi(|Q|^2 Q) \ .$$

Furthermore, there exists solutions of the form

$$u(t,x) = \alpha_1(t) Q\left(\frac{x - x_1(t)}{\kappa_1(t)}\right) + \alpha_2(t) Q\left(\frac{x - x_2(t)}{\kappa_2(t)}\right)$$

such that $\kappa_1(t) \to \lambda > 0 \ , \ \kappa_2(t) \simeq t^{-2} \ , \ t \to +\infty.$

Solitons for the focusing half–wave on the line

Krieger–Lenzmann–Raphaël (2013) found solitons for

$$i\partial_t u - |D|u + |u|^2 u = 0$$

by minimizing, for every velocity $\beta \in (-1, 1)$,

$$J_\beta(u) := \frac{\|u\|_{L^2}^2 ((|D| - \beta D)u, u)_{L^2}}{\|u\|_{L^4}^4}$$

Minimizers Q_β satisfy, after rescaling,

$$\frac{|D| - \beta D}{1 - \beta} Q_\beta + Q_\beta = |Q_\beta|^2 Q_\beta$$

so that the focusing half–wave equation is satisfied by

$$u_\beta(t, x) = \mathrm{e}^{it} Q_\beta \left(\frac{x - \beta t}{1 - \beta} \right)$$

The photonic limit of Q_β

Main observation : for $\beta^* < 1$ close enough to 1, there exists a smooth mapping $\beta \in (\beta^*, 1) \longmapsto Q_\beta \in H^\infty(\mathbb{R})$ such that

$$Q_\beta \underset{\beta \to 1}{\longrightarrow} Q$$

in H^s for every s.
Notice that

$$\|u_\beta\|_{L^2} \simeq \sqrt{1-\beta}\ .$$

Soliton interaction for the focusing half–wave

Theorem (PG, E. Lenzmann, O. Pocovnicu, P. Raphaël, 2015)

For every $\delta > 0, K > 0$, there exists $T > 0$ and a solution u of

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such that $\|u(0)\|_{H^1} \leq \delta \ , \ \forall t \geq T \ , \ \|u(t)\|_{H^1} \geq K \ .$

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Ansatz (Buslaev–Perelman, Merle, Martel, Raphaël, Krieger, Schlag, Tataru, ...)

$$u(t,x) = \sum_{j=1}^{2} \frac{e^{i\gamma_j(t)}}{\lambda_j^{\frac{1}{2}}(t)} Q_{\beta_j(t)} \left(\frac{x - x_j(t)}{\lambda_j(t)(1-\beta_j(t))} \right) + \varepsilon(t,x)$$

Perspectives

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- Random data for the cubic Szegő equation ? For the half–wave equation ?