Arnaud Debussche Lecture 1

Introduction to Stochastic PDEs

$$du = (Au + f(u))\,dt + \underbrace{\phi\, dw}$$

diff op nonlinearity additive noise

dw - white noise space-time

First some examples

Ex: Burgers: $A = \partial_{xx} u$, periodic + Dirichlet bdy con

$\phi = I$

Ex: KPZ: Same as Burger's except $\phi = \partial_x$

Very difficult problem

$\to \phi = |\partial_x|^\alpha$, α-small

Ex: Reaction Diffusion: $G \subset \mathbb{R}^d$ smooth bdd domain

$A = \Delta + b.c.$ $d=1$, $\phi = Id \to$ easy

$f(u) = -u^3 + \lambda u$ $d=2$, $\phi = Id \to$ no so easy

$d=3$, $\phi = Id \to$ MH 2014

Ex: Navier-Stokes: $G \subset \mathbb{R}^d$, $u(x,t) \in \mathbb{R}^d$, $x \in G$, $t > 0$

$$\begin{cases} du = (\nu \Delta u + (u \cdot \nabla) u + \nabla p)\,dt + \phi\, dw \\ \operatorname{div} u = 0 \\ bc \end{cases}$$

Other eqns:

- Same $\phi = \phi(u)$
- Nonlinear stochastic Schrödinger

$u \in \mathbb{C}$, $x \in \mathbb{R}^d$, $t > 0$

$i\,du = (\Delta u \pm |u|^\alpha u)\,dt + u \circ dW$

Brownian motion, Wiener processes

$(B(t))_{t\geq 0}$ is a B.M. if

$B(t) = B(t,\omega)$, $\omega \in \Omega$

where $(\Omega, \mathcal{F}, \mathbb{P})$ probability space

- $B(0)=0$
- $\forall \omega \in \Omega$, $t \longrightarrow B(t)$ is cts
- B is a Gaussian process, ie $\forall t_1, \dots, t_n \geq 0$, $(B(t_1), \dots, B(t_n))$ is a Gaussian vector.
- $B(t) \sim N(0,t)$
- $B(t) - B(s)$ indep of $B(s)$, $s \leq t$

  future is indep of present and past.
- $\mathbb{E}(|B(t) - B(s)|^2) = t-s \longrightarrow$ this implies B DNE in the classical sense

  $B \in C_t^{1/2-\varepsilon}$ a.s.

  $B \notin C^{1/2}$, $B \notin BV$ a.s.

Filtration: $(\mathcal{F}_t)_{t\geq 0}$ σ-field

$$\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F} \text{ for } s \leq t$$

X a random variable : X is $\mathcal{F}_t$ measurable if "X depends on the past up to t"

(ad) We ask: $\forall t$, $B(t)$ is $\mathcal{F}_t$-measurable.

Remarks:

- Formally, $(f_j)_{j\in\mathbb{N}}$ a Complete Orthonormal System (CONS) of $L^2(\mathbb{R}^+)$

$$\dot{B}(t,\omega) = \sum_j x_j(\omega) f_j(t). \quad x_j \text{ are indep } N(0,1)$$

$\rightarrow$ time white noise

- Formally: $\mathbb{E}(\dot{B}(t)\dot{B}(s)) = \delta_{t,s}$

$$\mathbb{E}\iint B(t)B(s)\dot{\varphi}(t)\dot{\psi}(s)\,dt\,ds = \mathbb{E}\int \varphi(t)\psi(t)\,dt$$

1) Wiener process: H a Hilbert space, a CONS $(e_k)_{k\in\mathbb{N}}$

$(B_k)_k$ a sequence of BM, indep, satisfying (ad) $\rightarrow$ they are adapted to $(\mathcal{F}_t)$

Let's say $H = L^2(G)$.

$$W = \sum_k \underbrace{B_k(t,\omega)\, e_k(x)}_{\text{function of } t,x,\omega} \quad \text{a cylindrical Weiner process in } H.$$

Problem:

$$\mathbb{E}(|W|_H^2) = \sum_k \mathbb{E}(|B_k(t)|^2) = \sum_k t = \infty$$

$H \hookrightarrow U$ a Hilbert space

$$\mathbb{E}(|W(t)|_U^2) = \sum_{k,l} \mathbb{E}(B_k(t) B_l(t))(e_k, e_l)$$

$$= \sum_k t |e_k|_H^2 < \infty \text{ iff } H \hookrightarrow U$$

Hilbert-Schmidt

Ex: $H = L^2(G)$, $U = H^s(G)$

$$H \xhookrightarrow[H.S.]{} U \quad \text{iff } s < -d/2$$

Remark: $\forall k$, $\dot{B}_k = \sum_j x_{jk} f_j \quad x_{jk}$ ind $N(0,1)$

$$\dot{W} = \sum_{j,k} f_j e_k$$

Space time white noise

Formally: $\mathbb{E}(\dot{W}(t,x)\dot{W}(s,y)) = \delta_{t,s}\,\delta_{x,y}$

We often smooth the noise.

Correlated noise: K-Hilbert space, $\varphi \in \mathcal{L}(H, K)$

$\varphi\, dW = \sum_k \varphi e_k\, dB_k$. φW lives in K iff

$\varphi \in \mathcal{L}_2(H, K)$

← Hilbert-Schmidt

2) Stochastic Integral

$\int_0^t \varphi(s)\, dW(s)$ where φ has values in $\mathcal{L}(H, W)$.

(e_k) - CONS of H, (f_j) - CONS of K

$$\int_0^t \varphi(s)\, dW(s) = \overbrace{\sum_{k\ell} \underbrace{\int_0^t (\varphi(s) e_k, f_\ell)\, dB_k}_{\text{Standard Ito integral in dim 1}} f_\ell}^{\text{convergence of a series in } K}$$

$$\varphi_{k,\ell}(s) = (\varphi(s) e_k, f_\ell), \quad \int_0^t \varphi_{k,\ell}(s)\, dB_k(s), \quad B_k \notin BV$$

Ito int $\approx \sum_i \varphi_{k,\ell} (B_k(t_{i+1}) - B_k(t_i))$

(t_i) subdivision of $[0, t]$

assume $\varphi_{k\ell}$ deterministic

$$\mathbb{E}\left(\int_0^t \varphi_{k,\ell}(s)\, dB_k(s)\right) = 0, \quad \mathbb{E}\left(\left(\int_0^t \varphi_{k,\ell}(s)\, dB_k(s)\right)^2\right) =$$

$$2 \sum_{ij} \varphi_{k\ell}(t_i) \varphi_{k\ell}(t_j) (B(t_{i+1}) - B(t_i)) (B(t_{j+1}) - B(t_j))$$

$$+ \sum \mathbb{E}\, \varphi_{k\ell}^2(t_i) (B(t_{i+1}) - B(t_i))^2$$

This is okay if φ is adaptive.

$$\varphi_{k,\ell} \in L^2(\Omega \times [0,t]) \xrightarrow[\text{isometry}]{} \int_0^t \varphi_{k,\ell}(s)\, dB_k \in L^2(\Omega)$$

if $\varphi_{k,\ell}$ adaptive $\longrightarrow$ in fact "predictable"

$$\mathbb{E}\left(\int_0^t \Phi(s)\,dW(s)\right) = 0,$$

$$\mathbb{E}\left(\left|\int_0^t \varphi(s)\,dW(s)\right|_K^2\right) = \sum_{\ell} \mathbb{E}\left|\sum_k \int_0^t (\phi(s)e_k, f_\ell)\,dB_k\right|^2$$

$$= \sum_{\ell,k} \mathbb{E}\left(\int_0^t (\varphi(s)e_k, f_\ell)\,dB_k\right)^2$$

(cross terms vanish)

$$= \sum_{\ell,k} \int_0^t \mathbb{E}(\varphi(s)e_k f_\ell)^2\,ds$$

$$= \mathbb{E}\int_0^t |\varphi(s)|^2_{\mathcal{L}_2(H,K)}\,ds$$

Remark: If $\underbrace{\int_0^t \varphi(s)\circ dB(s)}_{\text{Stratonovitch}} = \sum_i \varphi\left(\frac{t_i + t_{i+1}}{2}\right)\left(B(t_{i+1}) - B(t_i)\right)$

Not defined for as many processes, but more natural in physics.

3) Linear Heat Eqn

$\mathcal{O} \subset \mathbb{R}^d$ open bdd set on $[0,1]^d$

$A = \Delta + $ b.c.

W: cylindrical Wiener process on $H = L^2(\mathcal{O})$

Φ: Linear operator on H

$$\begin{cases} du = Au\,dt + \varphi\,dW \\ u(0) = u_0 \end{cases}$$

$\to$ Soln is $u(t) = e^{At}u_0 + \int_0^t e^{A(t-s)}\varphi\,dW$ in K

is the only weak soln (PDE sense) if the heat eqn is well-posed in K^*.

Question: What is the smoothness of u?

Ex: $\Phi = I$

u will live in $H^{2B}(G)$ iff $\int_0^t |e^{A(t-s)}|^2_{\mathcal{L}_2(H, H^{2B}(G))} ds < \infty$

$$|v|_{H^{2B}} \sim |(-A)^B v|$$

$$\int_0^t |(-A)^B e^{A(t-s)}|^2_{\mathcal{L}_2(H)} ds = \sum_k \int_0^t \lambda_k^{2B} e^{-2\lambda_k (t-s)} ds$$

where (λ_k) are eigenvalues of $-A$

$$= \sum_k \tfrac{1}{2} \lambda_k^{2B-1} (1 - e^{-2\lambda_k t})$$

it is known in many cases that $\lambda_k \sim c k^{2/d}$

$< \infty$

$\Leftarrow$ iff $\frac{2}{d}(2B-1) < -1$

$d=1 \Rightarrow B < 1/4 \rightarrow u$ has $1/2$-reg

$d=2 \Rightarrow B < 0 \rightarrow u$ has 0-reg

$d=3 \Rightarrow B < -1/4$

Case $d=1$; $G = (0,1)$, $u_0 = 0$.

$$u(t) = \int_0^t e^{A(t-s)} dW(s)$$

$$u(t,x) = \sum_k \int_0^t (e^{A(t-s)} e_k)(x) \, dB_k(s)$$

$$\mathbb{E}(|u(t,x) - u(s,y)|^2) \le c(|t-s|^{1/2} + |x-y|)$$

$e_k = \sin(kx)$

Thus $u \in C_t^{1/4-} C_x^{1/2-}$ a.s.

$$\mathbb{E}\left(|u|^p_{C_t^{1/4-} C_x^{1/2-}}\right) < \infty$$