Results and conjectures about extremal (almost) stable polynomials

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Let $p(x_1,...,x_m)$ be homogeneous polynomial of degree n with nonnegative coefficients, $p(1,1,...,1)=1$. Following **MSS**, define its matching univariate polynomial as $M_p(x) =: \prod_{1 \leq i \leq n}(I - \frac{\partial}{\partial x_i})p(x,x,...,x)$. There were several steps and revelations in the course of **MSS** proof of Kadison-Singer problem. But, IMHO, the main technical one was the sharp bound for stable polynomials p on the maximum root of $M_p(x)$, expressed in term of the gradient $a_i =: \frac{\partial}{\partial x_i}p(1,1,\ldots,1); 1 \leq m$. The follow up conjecture states the extremal stable polynomial, i.e. maximazing the maximum root of $M_p(x)$ given the gradient at the vector of all ones, is of rank one: e.g.
$p(x_1,...,x_n) = n^{-n}(a_1x_1 + \cdots + a_mx_m)^n$. The conjecture holds if, for instance, either $n=2$ or $m=2$.
I will review a general result of this type that makes the similar rank one conclusion for polynomials extremal respect to Van Der Waerden like bounds, e.g. homogeneous polynomials $p(x_1,...,x_n)$ of degree n such that

$$\frac{\partial^n}{\partial x_1,\ldots,\partial x_n}p(0,\ldots,0) = \frac{n!}{n^n} \inf_{x_i>0,\prod_{1\leq i\leq n} x_i=1} p(x_1,...,x_n).$$

The main result in this direction is rank one description of extremal strongly-log-concave polynomials(the Minkowski volume polynomial being the most interesting representative). Time permiiting, I will also describe an application of Kadison-Singer problem to the Quantum Linear Optics.