Tengren Zhang.
w/ Gye-Seon Lee.

Collar lemma for Hitchin reps

1) Background

Let S be a closed oriented surface, genus ≥ 2, $\Gamma = \pi_1(S)$

$\mathcal{T}(S) = \{$marked hyp str on $S\}$

$= \{$discrete faithful $\rho : \Gamma \longrightarrow PSL(2,\mathbb{R})\}/PSL(2,\mathbb{R})$

Fact: $\forall n \geq 2$, $\exists!$ (up to conj) irred $\iota_n : PSL(2,\mathbb{R}) \longrightarrow PSL(n,\mathbb{R})$

$\Rightarrow \iota_n : \mathcal{T}(S) \hookrightarrow \mathrm{Hom}(\Gamma, PSL(n,\mathbb{R}))/PSL(n,\mathbb{R}) =: \mathcal{X}_n(S)$

$\rho \longmapsto \iota_n \circ \rho$.

Def: $\mathrm{Hit}_n(S)$ is a component of $\mathcal{X}_n(S)$ that contains $\underbrace{\iota_n(\mathcal{T}(S))}_{\text{Fuchsian locus}}$

Thm (Labourie) $\forall \rho \in \mathrm{Hit}_n(S)$, ρ is discrete faithful and $\forall \gamma \in \Gamma$, $\rho(\gamma)$ is diagonalizable with distinct eigenvalues, $\lambda_n(\gamma) > \cdots > \lambda_1(\gamma)$

Def: $\forall \gamma \in \Gamma$, $\forall \rho \in \mathrm{Hit}_n(S)$, $\ell_\rho(\gamma) = \log\left(\frac{\lambda_n(\gamma)}{\lambda_1(\gamma)}\right)$

Example: • $\mathrm{Hitch}_2(S) = \mathcal{T}(S)$, $\ell_\rho(\gamma)$ is the hyp length of $[\gamma]$.

(Choi–Goldman): $\mathrm{Hitch}_3(S) = \mathcal{C}(S) := \{$real convex projective structure on $S\}$

$\ell_\rho(\gamma)$ is the Hilbert ~~metric~~ length of $[\gamma]$.

Def: • A convex $\mathbb{RP}^2$ surface is the quotient of a prop convex domain $\Omega \subseteq \mathbb{RP}^2$ by a $\overset{\text{sub}}{\text{gp}}$ of proj transf acting faithful prop disc and cocompactly on Ω.

• Hilbert metric Let $x, y \in \Omega$ $d_\Omega(x,y) = \log b(a,x,y,b)$.

descends to a metric on $\Gamma \backslash \Omega$.

- $\mathcal{C}(S) = \{(f, \Sigma)\ \ \Sigma$ is a convex $\mathbb{RP}^2$ surface, $f: S \to \Sigma$ diffeo$\} /_{\text{isot}}$

<u>Thm</u> (Benzeri, Kuiper)

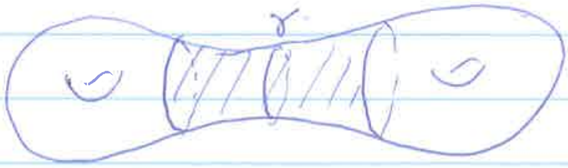
If $\Gamma \backslash \Omega$ is a convex $\mathbb{RP}^2$ surface, then Ω is strictly convex with C^1 boundary.

2) Main result:

<u>Classical collar lemma</u>: (Kean)

Let Σ be a hyperbolic surface. Let γ be a simple closed geodesic in Σ.

Then, $\cancel{A}N_\gamma := \{x \in \Sigma,\ d(x, \gamma) \le \sinh^{-1}\left(\frac{1}{\sinh(\frac{l(\gamma)}{2})}\right)\}$ is an embedding annulus.



Σ.

$\Rightarrow$ If η is a closed geod on Σ, then

$$\ell_\rho(\eta) \ge i(\eta, \gamma) \cdot 2 \cdot \sinh^{-1}\left(\frac{1}{\sinh \frac{l(\gamma)}{2}}\right)$$

$$I_\rho(\eta,\gamma) = \frac{1}{4} \cdot \exp\left(-\frac{l(\eta)}{2 i(\eta,\gamma)} - \frac{l(\gamma)}{2}\right)\left(\exp\left(\frac{l(\eta)}{i(\eta,\gamma)}\right) - 1\right)\left(\exp(l(\gamma) - 1)\right) \ge 1.$$

<u>Thm</u> (Lee, Z)

Let $\rho \in \mathrm{Hit}_n(S)$ let $\eta, \gamma \in \Gamma \setminus \{id\}$.

① $i(\eta, \gamma) \neq 0$, then $(\exp(\ell_\rho(\gamma)) - 1)(\exp(\ell_\rho(\eta)) - 1) \ge 1$

② $i(\eta, \gamma) \neq 0$ and γ is simple, then

$$\left(\exp \frac{\ell_\rho(\eta)}{i(\eta,\gamma)} - 1\right)\left(\exp \ell_\rho(\gamma) - 1\right) \ge 1$$

③ If γ is primitive and non-simple, then $\ell_\rho(\gamma) > \log 2$.

Rk: 1) Can improve the ineq if on allows dependence on n.

2) If $\rho(\Gamma)$ in $PSp(2k,\mathbb{R})$ or $PSO(K,K+1)$, have stronger ineq

3) Does not hold for all Anosov rep.

Ex: $QF(S) \underset{\text{Bers}}{\cong} \mathcal{T}(S) \times \mathcal{T}(S)$.

$\rho(\Gamma)\backslash \mathbb{H}^3 \underset{\text{top}}{\cong} S \times I$.

Fact (Bers) $\rho = (\rho_1, \rho_2)$, then $\forall \gamma \in \Gamma$, $\ell_\rho(\gamma) \leq 2\min\{\ell_{\rho_1}(\gamma), \ell_{\rho_2}(\gamma)\}$

Choose ~~two~~ γ_1 γ_2 has no intersection in S.
(s.c.c.)

Let $\{\rho_j^{(i)}\}_{i=1}^\infty$ be a sequence in $\mathcal{T}(S)$ correspond. to pinching γ_j

Let $\rho^{(i)} = (\rho_1^{(i)}, \rho_2^{(i)})$

$\lim_{i\to\infty} \ell_{\rho^{(i)}}(\gamma_1) = 0 = \lim_{i\to\infty} \ell_{\rho^{(i)}}(\gamma_2)$

4) Is the ineq. sharp?

$\exists$ $\{\rho_i\}_{i=1}^\infty$ in $\mathcal{T}(S)$ and $\{(\gamma_i, \eta_i)\}_{i=1}^\infty$ seq of pairs of closed curves s.t. $\lim_{i\to\infty} I_{\rho_i}(\eta_i, \gamma_i) = 1$

Conj: $\forall \rho \in Hit_n(S)$ $\exists \rho'$ in the Fuchsian locus s.t. $\forall \gamma \in \Gamma$,
$\ell_\rho(\gamma) \geq \ell_{\rho'}(\gamma)$.

5) (Vladimis – Yamelo)

Basmajian identity on $Hit_n(S)$

$\Rightarrow \forall \rho \in Hit_n(S)\ \forall \gamma, \eta \in \Gamma\ \ i(\eta,\gamma) \neq 0.\ \ \gamma$ simple.

$l_\rho(\gamma) \geq \underbrace{F(\rho(\eta))}_{\text{does not depend only on } l_\rho(\eta)}$

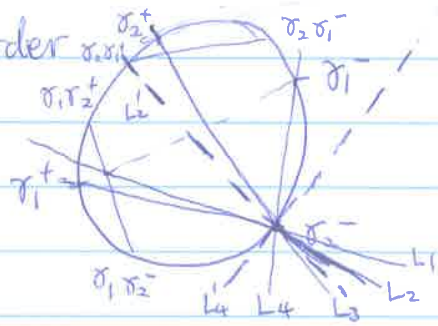
Corollary: Let $\rho \in Hit_n(S)$, then $\exists$ at most $3g-3$ primitive closed curves on S with length at most $\log(2)$.

Corollary: Let $M = SL_n(\mathbb{R})/SO(n)$ equipped with Riem metric.
M' — Hilbert —

Then inequality in thm hold with l_ρ replaced. $l_{\rho(\Gamma)\backslash M}$ or $l_{\rho(\Gamma)\backslash M'}$

3) Proof (point ①) in case when $n=3$.

Lemma: Let $\gamma_1\ \gamma_2 \in \Gamma$ s.t. $\gamma_1^+, \gamma_2^+, \gamma_1^-, \gamma_2^- \in \partial\Gamma$ in the order



Then $\gamma_1^+, \gamma_1 \cdot \gamma_2^+, \gamma_2 \cdot \gamma_1^+, \gamma_2^+ \in \partial\Gamma$ in the same order.

Choose $\rho \in Hit_3(S) \rightsquigarrow [f, \rho(\Gamma)\backslash\Omega]$

$\rightsquigarrow$ identification $\partial\Omega$ and $\partial\Gamma$.

$$\frac{\lambda_3(\gamma_1)}{\lambda_1(\gamma_1)} = C(L_1, L_2, L_3, L_4)$$
$$> C(L_1, L_2', L_3, L_4').$$
$$= \frac{\lambda_3(\gamma_2)}{\lambda_3(\gamma_2) - \lambda_2(\gamma_2)}$$

$$\Rightarrow \left(\frac{\lambda_3(\gamma_1)}{\lambda_1(\gamma_1)} - 1\right)\left(\frac{\lambda_3(\gamma_2)}{\lambda_1(\gamma_2)} - 1\right) \geq 1$$

<u>Thm</u> Let $\rho \in Hit_n(S)$, Let $\gamma_1\ \gamma_2 \in \Gamma$, $i(\gamma_1, \gamma_2) \neq 0$.

$$\frac{\lambda_n(\gamma_1)}{\lambda_1(\gamma_1)} \geq \frac{\lambda_{k+1}(\gamma_2)}{\lambda_{k+1}(\gamma_2) - \lambda_k(\gamma_2)} \quad \forall\ k = 1, \ldots, n-1.$$

<u>For general case</u>:

<u>Thm</u> (Guichard, Labourie)

$\rho \in \mathcal{X}_n(S)$, ρ is Hitchin $\Leftrightarrow \exists$ a ρ-equivariant Frenet curve $\xi : \partial\Gamma \to \cancel{\mathbb{RP}^n}\ \mathcal{F}(\mathbb{R}^n)$ (unique up to $PSL_n(\mathbb{R})$).