THIN GROUPS: ARITHMETIC AND GEOMETRIC VIEWPOINTS

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1. Introduction

Definition 1. If $\Gamma \leq \mathrm{GL}(n, \mathbb{Z})$ and G the Zariski closure of Γ in $\mathrm{GL}(n, \mathbb{C})$, we call Γ *thin* if $[G \cap \mathrm{GL}(n, \mathbb{Z}) : \Gamma] = \infty$.

1.1. **Apollonian circle packings.** Given 3 tangent circles, there exists a unique fourth target to all 3.

- Let a, b, c, d be the *curvatures* of four pairwise tangent circles. If $a, b, c, d \in \mathbb{Z}$, then all the curvatures in the packing will be integers.
- There exists infinitely many primitive integral Apollonian circle packings.

Theorem 1 (Descartes)**.** *If x_1, x_2, x_3, x_4 are the curvatures of four pairwise tangent circles, then*

$$Q(x_1, x_2, x_3, x_4) = 2 \sum x_i^2 - \left(\sum x_i \right)^2 = 0$$

a quadratic form of signature $(3, 1)$*.*

Where do thin groups come in? Consider quadruples of curvature of pairwise tangent circles. This corresponds to $A \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ in a packing containing (a, b, c, d). $A \leq O_Q(\mathbb{Z})$ is called an *Apollonian group* and is thin.

Let $\delta \sim 1.3056...$ be the Hausdorff dimension of the limit set of the packing.

Theorem 2 (Kontorovich-Oh)**.** *Let P be an Apollonian circle packing (ACP), then*

$$\#\{\textit{circles in a given bounded ACP of curvature } \leq X\} \sim c_P X^{\delta}$$

Theorem 3 (Bourgain-Fuchs)**.**

$$\#\{\textit{integers } < X \textit{ which are curvatures in a given ACP}\} \gg X$$

Conjecture 1 (Fuchs-Sanden, Graham-Lagarias-Mallows-Wilkes-Yan)**.** *Given a primitive integral ACP P, let P_{24} denote the residues mod 24 represented in P. Then there exists $X_P \in \mathbb{R}$ such that if $x \in \mathbb{Z}$, $x > X_P$ and $x \in P_{24}$, then x is the curvature of some circle in P.*

Theorem 4 (Bourgain-Kontorovich)**.** *This conjecture holds if one excludes a* 0*-density subset of* $\mathbb{Z}$*.*

Conjecture 2 (Fuchs-Sanden)**.**

$$\#\{\textit{circles of prime curvature } < X\} \sim \frac{cX^{\delta}}{\log X}$$

$c = 0.91...$

Theorem 5. *For a given primitive ACP P, let $\mathcal{O}(P)$ denote the corresponding orbit of A. Then*

$$Zcl\left(\left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in \mathcal{O}(P) \mid abcd \textit{ has at most 31 prime factors} \right\}\right) = Zcl(\mathcal{O}(P))$$

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2. The affine sieve

Problem 1. Given $\Gamma \subset \mathrm{GL}(n,\mathbb{Z})$, finitely generated, $\Gamma = \langle s \rangle$, $0 \neq v \in \mathbb{Z}^n$ and $f(x) \in \mathbb{Z}[x_1, \dots, x_n]$. We want to count points of $\mathcal{O}(f,r) = \{w \in \Gamma v \mid f(w) \text{ has at most } r \text{ prime factors}\}$

Theorem 6 (Bourgain-Gamburd-Sarnak, Salehi-Golsefidy-Sarnak)**.** *if Γ is "nice" then there exists $r \in \mathbb{Z}$ such that $Zcl(\mathcal{O}(f,r)) = Zcl(\Gamma v)$.*

What is this "niceness" property?

To each Γ we associate an infinite family of graphs $\{X_d\}_{d>1} \to X_d := \mathrm{Cay}(\Gamma/d, \overline{S})$. X_d is a finite connected graph and we can assume it is R-regular. We say Γ is "nice" if $\{X_d\}_{d>0}$ is an expander family.

Definition 2. Let $\{X_i\}_{i\geq 1}$ be an infinite family of finite, connected, R-regular graphs. Let M_I be the adjacency matrix of X_i. Denote the eigenvalues of M_i as $R = \lambda_0 > \lambda_1 \geq \cdots \geq \lambda_r$. $\{X_i\}_{i\geq 1}$ is an *expander family* if $\limsup_{n\to\infty} \lambda_1(M_n) < R$.

Remark 1. This means that there is a spectral gap, the bigger the spectral gap, the nicer your group is. Unfortunately in the case of most thin groups, using the affine sieve does not give us bounds for the spectral gap. Contrast this to non thin groups, where if Γ is a congruence subgroup of $\mathrm{SL}(2,\mathbb{Z})$, then Selberg's $\frac{3}{16}$ theorem gives a lower bound on the spectral gap for γ.

Question 1. Given a finitely generated subgroup of $\mathrm{GL}(n,\mathbb{Z})$, can you tell if it is thin?

The definition of thinness has something to do with the index inside the Zariski closure. Generally finding the Zariski closure is not the difficult part, but determining its index is hard:

Fact 1. *Given a finitely generated subgroup of $GL(n,\mathbb{Z})$, the question of whether or not it has finite or infinite index is undecidable.*

2.1. **Fuchs-Meiri-Sarnak.** Taking the setup from Beukers-Heckman, '89:

Assume $\alpha_i, \beta_i \in \mathbb{Q}, \in [0,1]$ with $\alpha_i \neq \beta_i$. Let $\Theta = z\frac{\mathrm{d}}{\mathrm{d}z}$

$$D(\alpha,\beta) = D(\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_n) = \prod_{i=1}^{n}(\Theta + \beta_i - 1) - z\prod_{i=1}^{n}(\Theta + \alpha_i)$$

The equation $Du = 0$ is regular outside of $\{0,1,\infty\}$ in $\mathbb{P}^1(\mathbb{C})$. $D(\alpha,\beta)$ gives rise to a monodromy representation of $\pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \{0,1,\infty\}, z_0)$ in $\mathrm{GL}(n,\mathbb{C})$. We denote the monodromy group by $H(\alpha,\beta) = \langle A, B\rangle$.

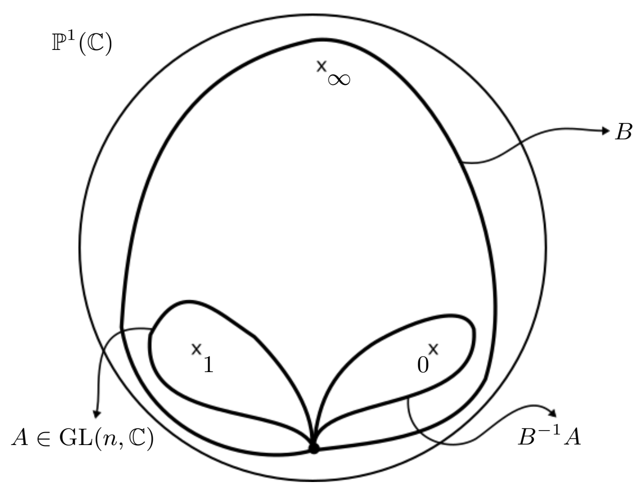

Figure 1. The generators of the monodromy group

Theorem 7 (Levelt)**.** *Let* $\prod_{j=1}^{n}(x - e^{2\pi i\alpha_j}) = a_n x^n + \cdots + a_0)$*,* $\prod_{j=1}^{n}(x - e^{2\pi i\beta_j} = b_n x^n + \cdots + b_0$*, then* $H(\alpha, \beta) = \langle A, B\rangle$ *where*

$$A = \begin{pmatrix} 0 & \cdots & 0 & -a_0 \\ 1 & & 0 & -a_1 \\ & \ddots & & \vdots \\ 0 & & 1 & -a_n \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & \cdots & 0 & -b_0 \\ 1 & & 0 & -b_1 \\ & \ddots & & \vdots \\ 0 & & 1 & -b_n \end{pmatrix}.$$

Fact 2 (Beukers-Heckman)**.** $H(\alpha, \beta) \leq GL(n, \mathbb{Z})$ *if an only if the characteristic polynomials of* A, B *are products of cyclotomes.*

We consider the case where $H(\alpha, \beta) \leq \mathcal{O}_f(\mathbb{Z})$ where f has signature $(n-1, 1)$.

Theorem 8 (Fuchs-Meiri-Sarnak)**.** *There are 7 infinite families of thin hyperbolic hypergeometric monodromy groups.*

Consider $C = B^{-1}A$. Then $C^2 = I$ and C has eigenvalue -1 with multiplicity 1, 1 with multiplicity $n-1$. Consider

$$H_r = \langle B^i C B^{-i} \mid i \in \mathbb{Z}\rangle$$

if H_r, is thin then so is $H(\alpha, \beta)$. C looks like it is a reflection in a hyperplane, due to the eigenvalues, so a theorem of Vindberg says that a group generated by those cannot be arithmetic (so must be thin). However, C is not a reflection and in fact, does not act isometrically on hyperbolic space. So instead take $-C$, the is a Cartan involution, and can then tie H_r to a group generated by reflections.

Question 2. Can we come up with other methods, perhaps geometric, outside of hyperbolic geometry, to do the same thing?