GEOMETRIC AND ARITHMETIC ASPECTS OF BOUNDED ORBITS ON HOMOGENEOUS SPACES

DMITRY KLEINBOCK

Recall: Ratner's Theorems were motivated by Raghunathan's Conjecture which is connected to Oppenheim Conjecture which states that every bounded orbit is close.

1. Actions of 1-parameter subgroups on G/Γ

Let $G = \mathrm{SL}(3, \mathbb{R})$ and $\Gamma = \mathrm{SL}(3, \mathbb{Z})$.

Theorem 1 (Ratner)**.** *If $U = \{u_x\} \subset G$ is a unipotent subgroup then $\overline{Ux}$ is algebraic, $\overline{Ux} = Lx, and, if x = g\Gamma$,$g^{-1}\Gamma g \cup L$ is a lattice in L. If $L \neq G$, this happens rarely.*

Take $F = \{g_t\} = \begin{pmatrix} e^{t/2} & & \\ & e^{t/2} & \\ & & e^{-t} \end{pmatrix} \subset G$, then $F_+ = \{g_t \mid t \geq 0\}$.

For $\alpha, \beta \in \mathbb{R}$ we define $u_{\alpha,\beta} = \begin{pmatrix} 1 & & \alpha \\ & 1 & \beta \\ & & 1 \end{pmatrix}$.

Definition 1. (α, β) called *badly approximable*, $(\alpha, \beta) \in BA$, if there exists $c > 0$ such that $\max(|q\alpha - p|, |q\beta - r|) > \frac{c}{q^{1/2}}$, for all $p, r, q \in \mathbb{Z}, q \neq 0$.

Theorem 2 (Dani, 1985)**.** *$F_t u_{\alpha,\beta}\Gamma$ is bounded if and only if $(\alpha, \beta) \in BA$*

Fact 1 (W. Schmidt, '66)**.** *T The set of badly approximable vectors is a set of measure 0 and Hausdorff dimension 2.*

Corollary 1 (Dani)**.** $\dim\left(\{x \in G/\Gamma \mid F_+x \textit{ is bounded}\}\right) = \dim(G/\Gamma)$

This is big, in contrast to the unipotent case where it is small.

Remark 1. $H = \{u_{\alpha,\beta}\}$ is the unstable horosphere with respect to F_+. Thus to study bounded orbits, it is enough to study this group, H.

Theorem 3 (Dani, '86)**.** *If the $\mathbb{R}$-rank$(G) = 1$, then the set of bounded orbits has full dimension.*

Conjecture 1 (Margulis, '90)**.** *if F is "not unipotent" then* $\dim(\{x \in G/\Gamma \mid Fx \textit{ is bounded}\}) = \dim(G/\Gamma)$, *vol*$(G/\Gamma) < \infty$.

Theorem 4. *Proved by Kleinbock-Margulis in '96 using exponential mixing.*

Proof. Given a surface that stretches to infinity we would like to construct a bounded orbit. We start with a point in the unstable foliation, apply g_t with t large, then the image of the piece of the horocycle will equidistribute. Fix a compact set, then most of the image will lie inside the compact set. Chop the image into pieces similar to the one you started with, repeat the process inside this new piece, and after iterating you have constructed a Cantor set of bounded orbits. If you move the boundary of this compact set towards infinity you get a bigger Cantor set and the dimension will go to the full dimension. □

Question 1. Is there a way to connect this picture with number theory?

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Example 1. Take $G = \mathrm{SL}(3,\mathbb{R})$, $\Gamma = \mathrm{SL}(3,\mathbb{Z})$, $F^{\lambda,\mu} = \left\{ \begin{pmatrix} e^{\lambda t} & & \\ & e^{\mu t} & \\ & & e^{-t} \end{pmatrix} \right\}$ where $\lambda + \mu = 1$. Here the unstable horosphere is given by $\left\{ \begin{pmatrix} 1 & \gamma & \alpha \\ & 1 & \beta \\ & & 1 \end{pmatrix} \right\}$, which means the theorem cannot be applied in this case.

Proposition 1. *$F_+^{\lambda,\mu} u_{\alpha,\beta}\Gamma$ is bounded if and only if $(\alpha,\beta) \in BA_{\lambda,\mu}$, that is, there exists a $c > 0$ such that for all $p, q, r \in \mathbb{Z}$, $q \neq 0$, we have $\max(q^{-\lambda}|q\alpha - p|, q^{\mu}|q\beta - r|) > c$.*

Schmidt's result gives a winning property of BA.

Conjecture 2 (Schmidt)**.** *$BA_{1/3,2/3} \cap BA_{2/3,1/3} \neq \emptyset$*

Remark 2. If $(\alpha,\beta) \neq BA_{\lambda,\mu}$ then for all c there exist solutions to $q|q\alpha - p||q\beta - r| < c^2$

Theorem 5. *The conjecture was proven by Badziahin-Pollington-Velani in 2011.*

Theorem 6 (An)**.** *$BA_{\lambda,\mu}$ is α_0-winning for all λ, μ.*

2. A tutorial on Schmidt games

A Schmidt game is played with two players, Alice and Bob and each are assigned a real number α and β respectively. We play on a metric space X with a target set S. Bob goes first a chooses a ball B_1 with radius r, Alice then chooses a ball, A_1, with radius αr, Bob then chooses a ball, B_2, inside A_1 with radius $\beta\alpha r$, play continues in this manner. Alice wins if $\bigcap B_i \in S$, if this happens S is called (α,β)-winning.

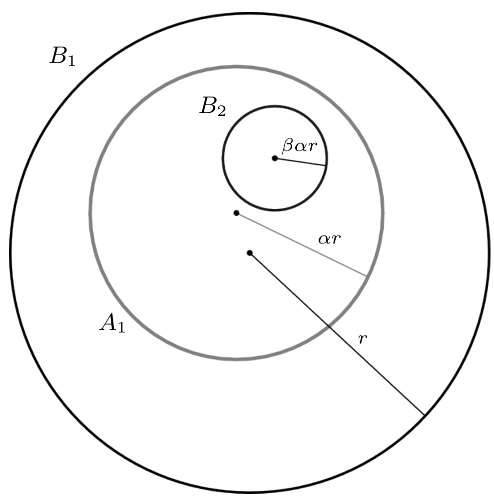

Figure 1. The first three steps in a Schmidt game

Definition 2. S is *α-winning* if and only if S is (α,β)-winning for all $\beta > 0$.

Schmidt proved several properties of these games:

- α-winning implies full Hausdorff dimension
- If each S_i is α-winning then so is $\bigcap_{i=1}^{\infty} S_i$

Corollary 2. *$\bigcap_{i=1}^{\infty} BA_{\lambda_i,\mu_i}$ has dimension 2 for all (λ_i,μ_i).*

We would like to have a dynamical understanding of this.

This corollary implies that $\{x \in G/\Gamma \mid F_i^+ x \text{ is bounded } \forall i\} \neq \emptyset$.

3. Recent Developments

Define $E(F) = \{x \in G/\Gamma \mid Fx \text{ is bounded}\}$.

Theorem 7 (An-Guan-Kleinbock, '15). *If $G = SL(3,\mathbb{R})$, $\Gamma = SL(3,\mathbb{Z})$, then for all diagonalizable $\{F_i^+\} \subset G$,*

$$\dim\left(\bigcap_{i=1}^{\infty} E\left(F_i^+\right)\right) = \dim(G/\Gamma)$$

As a consequence of Thm. 7, we get

Theorem 8. *$E(F^+)$ is a winning set of a Hyperblane Absolute Game.*

4. A tutorial on Hyperbolic Aboslute Games

We again have two players, Alice and Bob, and Bob is assigned a real number β. We play on a homogeneous space. We set $\varepsilon \le \beta, \delta \ge \beta$ and $\beta < \frac{1}{3}$. Bob again starts by choosing a ball B_1 of radius r, Alice then chooses a hyperplane L_1 and cuts out an εr-neighborhood around it, Bob next chooses a ball B_2 of radius δr, play continues in this manner. Alice wins if $\bigcap B_i \in S$, and S is called β-hyperplane absolute winning (HAW).

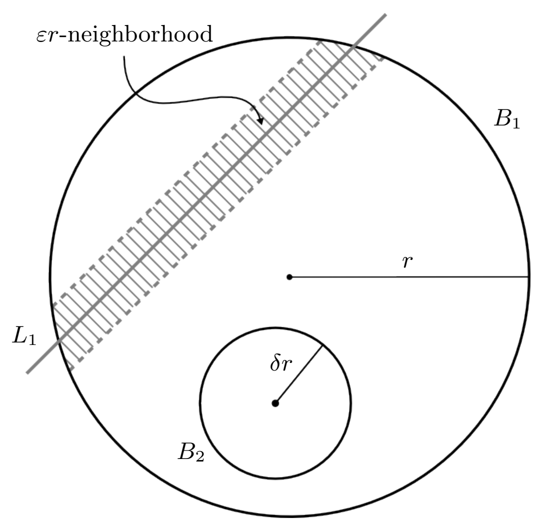

Figure 2. The first three steps in a hyperbolic absolute game

Definition 3. S is *HAW* if and only if S is β-winning for all $\beta > 0$.

Schmidt proved several properties of these games:

- HAW implies full Hausdorff dimension
- If each S_i is HAW then so is $\bigcap_{i=1}^{\infty} S_i$
- HAW is invariant under C^1 maps

The last property means that Alice does not have to cut hyperplanes, but she can also remove hypersurfaces, and also if M is your smooth manifold, you can define charts and play the game inside any of these charts.

Theorem 9 (BFKRW). *BA is HAW*

Theorem 10 (Nesharim-Simmons). *$BA_{\lambda,\mu}$ is HAW*

Theorem 11 (AGK). $E(F_{\lambda,\mu}^+) \cap \left\{\begin{pmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{pmatrix}\right\}\Gamma$ *is HAW*

Conjecture 3. *$E(F^+)$ is HAW if F^+ is diagonalizable.*