THE WEYL CHAMBER FLOW AND APPLICATIONS

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1. More details on the proof of last theorem

Theorem 1. *Let $\Gamma < G$ be a cocompact lattice, G a semi simple Lie group of rank 1, $G \neq SO(2n,1)$, then Γ contains a surface subgroup.*

1.1. Ergodic framework.

Let $M = \Gamma \backslash G/K$.

Definition 1. We say an action of a Lie group L on a Riemannan manifold M is *Anosov* if the following properties hold:

- The orbits of the action foliate M
- There exists $a \in L$ and an a-invariant splitting of the tangent bundle
$$TM = E^0 \oplus E^+ \oplus E^-$$
such that for all $v \in E^{\pm}$ we have
$$||\mathrm{d}a^{\mp}(x)|| < e^{-\kappa}||x||$$
- The growth of vectors in E^0 is subexponential

If $L = \mathbb{R}$ we get an *Anosov flow.*

Example 1. Consider the action of $\mathrm{SL}(2,\mathbb{R})$ on itself. $X = \begin{pmatrix} t & \\ & -t \end{pmatrix}$, then $E^0 = \mathrm{Span}\{X\}$, $E^- = \mathrm{Span}\left\{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right\}$ and $E^+ = \mathrm{Span}\left\{\begin{pmatrix} 0 & \\ 1 & 0 \end{pmatrix}\right\}$

Definition 2. A sequence of orbit segments of length $\geq R$ with jumps of size at most ϵ is called an (R,ϵ)-*pseudo-orbit.* If the Huasdorff distance between an orbit and the sequence is less than δ we call it δ-*close* to the orbit.

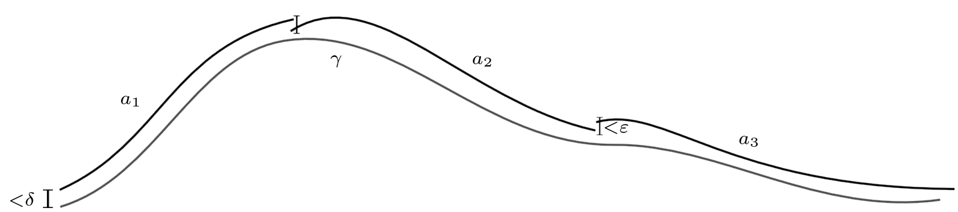

Figure 1. A sequence, a_1, a_2, a_3, of orbit segments which form a (R,ϵ)-pseudo-orbit that is δ-close to the orbit γ

Lemma 1 (Shadowing Lemma)**.** *Consider an Anosov flow on a compact manifold M. Then for all $\delta > 0$ there exists an $R > 0$ and $\epsilon > 0$ such that any (R,ϵ)-pseudo-orbit is δ-close to an orbit.*

Lemma 2 (Closing Lemma)**.** *Let M be a closed manifold with negative sectional curvature and $\tilde{M}$ its universal cover. Then every approximately-closed pseudo-orbit is close to a periodic geodesic.*

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Proof. If ε is less than the injectivity radius we can lift each orbit segment to a geodesic segment in $\tilde{M}$. Look at the orbit of the starting point x, since the deck transformation group is cocompact, there will be an image point φx in a uniformly bounded neighborhood of the ending point. The unique geodesic that connects x and φx projects to a closed object with a corner in the projection manifold. If you have some geometric control over the angles in the universal cover, it is possible to compare the outgoing vector to the starting vector, and if these are close, then you will be near a periodic geodesic, and you can find one in the quotient manifold.

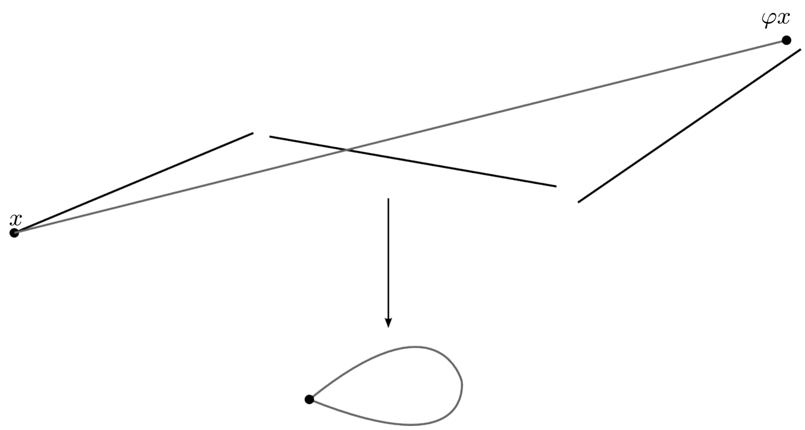

FIGURE 2. The projection of the geodesic connecting x and φx

□

Recall that a tripod is a triple of points (v_1, v_2, v_3) at pairwise angle $\frac{2\pi}{3}$ in the tangent plane of a totally geodesic hyperbolic plane in M.

Definition 3. Let S be a hyperbolic surface. Then for $\delta \in (0, 1/4)$, and $R > 100$ a pants decomposition of S is called (R, δ)-*tight* if the following properties hold:

- the lengths of the pants curves is in $[R - \delta, R + \delta]$
- seams are aligned with shear in $[1 - \delta, 1 + \delta]$

Recall the construction from before where we glued two truncated hyperbolic triangles to form a pair of pants. If we can align the pairs of pants and glue them to form a (R, δ)-tight pants decomposition of the surface, for some $R \gg 100$ and $\delta \ll 0$, then the fundamental group of the surface will inject into the fundamental group of the manifold and required.

1.2. **Mixing condition.** To satisfy the above geometric property, we have to adjust the gluing such that the foot-points and directions of the seams on the same geodesic satisfy the following properies:

- The distance between two foot-points should be roughly one
- Let v and w be the directions of two seams on the same geodesic, then moving v by parallel transport to the foot-point of w and reversing its direction should be approximately w with error $\sim \frac{1}{R^2}$

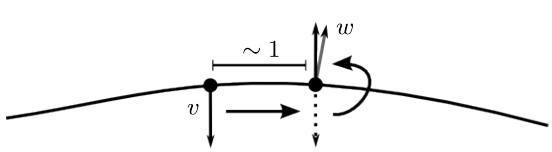

FIGURE 3. The condition on the directions of the seams

The error of $\frac{1}{R^2}$ corresponds to polynomial mixing of rate 2. To ensure that we can do this, we project the measure on tripods to the boundary geodesics which gives a measure on the bundle of unit vectors over a periodic geodesic γ, each fiber consists of the unit vector and the vector tangent to γ which span a totally real frame. Normalize to get a measure $f\mu$ where μ is standard Lebesgue measure and f has values in $\left[1-\frac{c}{R^2}, 1+\frac{c}{R^2}\right]$. This allows us to find a diffeomorphism φ of the bundle, where $\varphi^*(f\mu) = \mu$ and $d(x, \phi(x)) < \frac{c'}{R^2}$.

2. Higher rank

Let $G = \mathrm{SL}(3, \mathbb{R})$ a semisimple Lie group of rank $= 2$, $\Gamma < G$ torsion free and cocompact, and $K = \mathrm{SO}(3)$. Let $\mathfrak{k}$ be the Lie algebra of K, $\mathfrak{g}$ the Lie algebra of G, then we have the standard decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{p}$ is the orthogonal complement of $\mathfrak{k}$ with respect to the killing form. Take $\mathfrak{a} \subset \mathfrak{p}$ to be a maximal abelian subalgebra.

2.1. Root decomposition.

We have a *root system* $\Lambda \subset \mathrm{Hom}(\mathfrak{a}, \mathbb{R})$ such that

$$\mathfrak{g} = \mathfrak{a} \oplus \sum_{\lambda \in \Lambda} \mathfrak{g}_\lambda$$

Definition 4. *Walls* in $\mathfrak{a}$ are hyperplanes annihilated by some $\lambda \in \Lambda$, the components of the complement are the *Weyl chambers.*

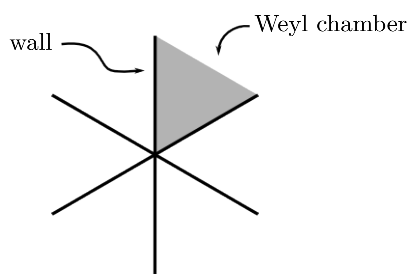

Figure 4. Walls and Weyl chambers

Let

$$A = \left\{ \begin{pmatrix} e^t & & \\ & e^s & \\ & & e^{-s-t} \end{pmatrix} \mid s, t \in \mathbb{R} \right\},$$

an abelian subgroup of $\mathrm{SL}(3, \mathbb{R})$.

Theorem 2 (Im Hof '85). *The action of the Lie subgroup,* $\exp \mathfrak{a} = A$*, by right translation defines an Anosov action on* $\Gamma \backslash SL(3, \mathbb{R})$.

Proof. Let $\Lambda^+ = \{\lambda(\log a) > 0\}$ and $\Lambda^- = \{\lambda(\log a) < 0\}$, where a is chosen to be $\begin{pmatrix} e^1 & & \\ & e & \\ & & e^{-2} \end{pmatrix}$, for example. We define $\mu^+ = \sum_{\lambda \in \Lambda^+} \mathfrak{g}_\lambda$, $\mu^- = \sum_{\lambda \in \Lambda^-} \mathfrak{g}_\lambda$, $\xi = \sum_{\lambda \in \Lambda^+} \xi_\lambda$, then we get the following decomposition $\mathrm{d}a(\xi) = \sum_{\lambda \in \Lambda^+} e^{-\lambda(\log a)} \xi_\lambda + \mu^+$. ∎

2.2. Generalizing.

Replace geodesic flow by the action of the element in the center of the Weyl chamber. Replace the totally real frames by frames defined by vectors in the sum of these to root spaces. Exponential mixing is not needed, we can use symmetries, but we do need a definite rate of mixing, at least polynomial. We may be able to generalize to $\mathrm{SL}(n, \mathbb{R})$, but there seems to be obstacles in trying to generalize to irreducible lattices in semisimple groups which are not simple.