Entropy in the cusp and singular systems of linear forms

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Singular systems

On Euclidean space we fix the maximum norm $\|\cdot\|$.

Theorem (Dirichlet's theorem for linear forms)

Let $m, n \in \mathbb{N}$ be given. For any $m \times n$ real matrix s and any $N \in \mathbb{N}$ there exist $\mathbf{q} \in \mathbb{Z}^n$ and $\mathbf{p} \in \mathbb{Z}^m$ such that

$$\|s\mathbf{q} - \mathbf{p}\| < \frac{1}{N^{n/m}} \text{ and } 0 < \|\mathbf{q}\| < N.$$

We say that the matrix $s \in M_{m,n}$ is a **singular system of m linear forms in n variables** if for any $\epsilon > 0$ there exists $N_0 \in \mathbb{N}$ such that for any $N > N_0$ there exist $\mathbf{q} \in \mathbb{Z}^n$ and $\mathbf{p} \in \mathbb{Z}^m$ such that

$$\|s\mathbf{q} - \mathbf{p}\| < \frac{\epsilon}{N^{n/m}} \text{ and } 0 < \|\mathbf{q}\| < N.$$

Singular systems

Theorem (KKLM'14)

The Hausdorff dimension of the set of $s \in M_{m,n}$ which are singular is at most $mn - \frac{mn}{m+n}$.

When $m + n = 3$ it was shown by Y. Cheung (2011) that the Hausdorff dimension of the set of singular pairs is $4/3$. Recently Y. Cheung and N. Chevallier (2014) showed that the set of singular m-vectors ($n = 1$) has Hausdorff dimension $m - \frac{m}{m+1}$.

Conjecture. The above theorem is sharp for $m + n \geq 3$.

Dani's correspondence

Fix any $t > 0$ and consider the dynamical system (X_{m+n}, g_t), with the action given by

$$g_t \cdot x = g_t x,$$

where

$$X_{m+n} := SL(m+n, \mathbb{R})/SL(m+n, \mathbb{Z})$$

and

$$g_t := diag(e^{nt}, ..., e^{nt}, e^{-mt}, ..., e^{-mt}).$$

The *unstable horospherical subgroup* U with respect to g_t can be identified with the space $M_{m,n}$ of $m \times n$ real matrices:

$U = \{u_s : s \in M_{m,n}\}$ where $u_s := \begin{pmatrix} I_m & s \\ 0 & I_n \end{pmatrix}$.

Dani's correspondence: s is a singular system of m linear forms in n variables if and only if $g_{\ell t} u_s SL(m+n, \mathbb{Z}) \to \infty$ in X_{m+n} as $\ell \to \infty$.

Entropy in the cusp

Theorem (KKLM'14)

For every sufficiently large integer $t > 0$ there exists a compact subset $Q = Q(t)$ of X_{m+n} such that

$$h_\mu(g_1) \le \big(m + n - 1 + \mu(Q)\big)\, mn + \frac{3 \log t}{t},$$

for any g_1-invariant probability measure μ on X_{m+n}.

Corollary

For any $h > 0$ and any sequence $(\mu_k)_{k\ge 1}$ of g_1-invariant probability measures on X_{m+n} with entropies $h_{\mu_k}(g_1) \ge h$, any weak limit μ of the sequence satisfies*

$$\mu(X_{m+n}) \ge \frac{h}{mn} - (m + n - 1)\,.$$

Entropy in the cusp

Conjecture. For any constant $h \in \big[0, (m+n)mn\big]$ there should exists a sequence of probability invariant measures $(\mu_k)_{k\geq 1}$ with $\lim_k h_{\mu_k}(g_1) = h$ such that the limit measure μ satisfies

$$\mu(X_{m+n}) = \max\left\{\frac{h}{mn} - (m+n) + 1,\ 0\right\}$$

It is known to be true when $\min(m,n) = 1$, (K.'11).

To prove the main results

The main results follow from the following.

Theorem

For sufficiently large t there are 'nice' compact sets Q in X_{m+n} such that for any $x \in Q$, $N \in \mathbb{N}$, and $\delta \in (0,1)$ the set

$$\left\{ u \in B_1^U : \frac{1}{N} \big| \{ \ell \in \{1, \ldots, N\} : g_{\ell t} u x \notin Q \} \big| \geq \delta \right\}$$

can be covered with $t^{3N} e^{(m+n-\delta)mntN}$ balls in U of radius $e^{-(m+n)tN}$.

Main idea

Method of the proof is based on integral inequalities developed by A. Eskin, G.A. Margulis and S. Mozes (1998). They show that there exists a positive continuous height function $\alpha : X_{m+n} \to \mathbb{R}$ with $\alpha(x) \to \infty$ as $x \to \infty$ in X_{m+n} and a constant $c > 0$ such that

$$\int_K \alpha(g_t kx)dk \leq c\alpha(x) + B,$$

for some constant B, where $K = SO(m) \times SO(n)$.
We show that there exists a positive continuous height function $\alpha : X_{m+n} \to \mathbb{R}$ with $\alpha(x) \to \infty$ as $x \to \infty$ in X_{m+n} such that

$$\int_K \alpha(g_t kx)dk \leq t^2 e^{-mnt}\alpha(x) + B,$$

for some constant B, where $K = SO(m+n)$.

Height function

Step 1. (Choosing exponents, β_i's) For any $i \in \{1, \ldots, m+n-1\}$ and decomposable $v \in \bigwedge^i \mathbb{R}^{m+n}$

$$\int_K \|g_t k v\|^{-\beta_i} \, dk \le t^2 e^{-mnt} \|v\|^{-\beta_i},$$

where $\beta_i = \frac{m}{i}$ if $i \le m$ and $\beta_i = \frac{n}{m+n-i}$ if $i > m$.

Height function

Step 2. (Choosing weights, ω_i's) For any $i = 1, \ldots, m+n$ and any $x \in X_{m+n}$ we let $F_i(x)$ denote the set of all i-dimensional subgroups of x. For any $L \in F_i(x)$ we let $\|L\|$ denote the volume of $L/(L \cap x)$. We define

$$\alpha_i(x) := \max \left\{ \frac{1}{\|L\|} : L \in F_i(x) \right\}.$$

Clearly, $\alpha_{m+n}(x) = 1$ and for convenience let $\alpha_0(x) := 1$ for all $x \in X_{m+n}$. For a fixed $t > 0$, there exist constants $\omega_0, \ldots, \omega_{m+n}$ such that

$$\int_K \alpha(g_t kx)\, dk \leq t^2 e^{-mnt} \alpha(x),$$

for $\alpha(x)$ large, where

$$\alpha := \sum_{i=0}^{m+n} \omega_\ell \alpha_\ell^{\beta_i}.$$

Thank You!