Quantum ergodicity

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- Quantum ergodicity on manifolds (comparing < 0 curvature, > 0 curvature and 0 curvature).
- QE on large regular graphs.

M a compact riemannian manifold, of dimension d.

$$\Delta\psi_k = -\lambda_k\psi_k$$

$$\|\psi_k\|_{L^2(M)} = 1$$

in the limit $\lambda_k \longrightarrow +\infty$.

We study the weak limits of the probability measures on M,

$$|\psi_k(x)|^2 d\mathrm{Vol}(x)$$

$\lambda_k \longrightarrow +\infty$.

This question is linked with the ergodic theory for the geodesic flow / billiard flow.
Hence the name quantum ergodicity.

Let $(\psi_k)_{k\in\mathbb{N}}$ be an orthonormal basis of $L^2(M)$, with

$$-\Delta\psi_k = \lambda_k\psi_k, \qquad \lambda_k \leq \lambda_{k+1}.$$

QE theorem (simplified) :

Theorem (Shnirelman, Zelditch, Colin de Verdière)

Assume that the action of the geodesic flow is **ergodic** *for the Liouville measure. Let $a \in C^0(M)$. Then*

$$\frac{1}{N(\lambda)} \sum_{\lambda_k \leq \lambda} \left| \int_M a(x)|\psi_k(x)|^2 d\mathrm{Vol}(x) - \int_M a(x) d\mathrm{Vol}(x) \right|^2 \longrightarrow 0.$$

Equivalently, there exists a subset $\mathcal{S} \subset \mathbb{N}$ of density 1, such that

$$\int_M a(x)|\psi_k(x)|^2 d\mathrm{Vol}(x) \xrightarrow[k \longrightarrow +\infty]{k \in \mathcal{S}} \int_M a(x) d\mathrm{Vol}(x).$$

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$$|\psi_k(x)|^2 d\mathrm{Vol}(x) \xrightarrow[k\longrightarrow+\infty]{k\in\mathcal{S}} d\mathrm{Vol}(x)$$

in the weak topology.

The full statement uses analysis on phase space, i.e. $T^*M = \{(x, \xi), x \in M, \xi \in T^*_x M\}$.

For $a = a(x, \xi)$ a "reasonable" function on T^*M, we can define an operator on $L^2(M)$,

$$a(x, D_x)$$

Say $a \in \mathcal{S}^0(T^*M)$ if a is smooth and 0-homogeneous in ξ (i.e. a is a smooth fn on the sphere bundle).

$$-\Delta\psi_k = \lambda_k \psi_k, \qquad \lambda_k \leq \lambda_{k+1}.$$

For $a \in \mathcal{S}^0(T^*M)$, we consider

$$\langle \psi_k, a(x, D_x)\psi_k \rangle_{L^2(M)}.$$

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For $a \in \mathcal{S}^0(T^*M)$, we consider

$$\langle \psi_k, a(x, D_x)\psi_k \rangle_{L^2(M)}.$$

This amounts to $\int_M a(x)|\psi_k(x)|^2 d\mathrm{Vol}(x)$ if $a = a(x)$.

Let $(\psi_k)_{k\in\mathbb{N}}$ be an orthonormal basis of $L^2(M)$, with

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QE theorem :

Theorem (Shnirelman, Zelditch, Colin de Verdière)

Assume that the action of the geodesic flow is **ergodic** *for the Liouville measure. Let $a(x,\xi) \in \mathcal{S}^0(T^*M)$. Then*

$$\frac{1}{N(\lambda)} \sum_{\lambda_k \leq \lambda} \left| \langle \psi_k, a(x, D_x)\psi_k\rangle_{L^2(M)} - \int_{|\xi|=1} a(x,\xi)dxd\xi \right|^2 \longrightarrow 0.$$

Idea of proof.

For any bounded operator K on $L^2(M)$, define the quantum variance

$$Var_\lambda(K) = \frac{1}{N(\lambda)} \sum_{\lambda_k \leq \lambda} \left| \langle \psi_k, K\psi_k \rangle_{L^2(M)} \right|^2 .$$

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The proof start from the trivial observation that

$$Var_\lambda([\sqrt{-\Delta}, K]) = 0$$

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In addition, if $K = a(x, D_x)$ is a pseudodifferential operator with $a \in \mathcal{S}^0(T^*M)$, then

$$[\sqrt{-\Delta}, a(x, D_x)] = (Xa)(x, D_x) + b(x, D_x)$$

where b is -1-homogeneous in ξ and X is the derivation along the geodesic flow.

This implies that

$$Var_{\lambda}((Xa)(x, D_x)) \underset{\lambda \longrightarrow +\infty}{\longrightarrow} 0.$$

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$$Var_\lambda(a(x, D_x)) \leq C \int_{|\xi|=1} |a(x,\xi)|^2 dx d\xi.$$

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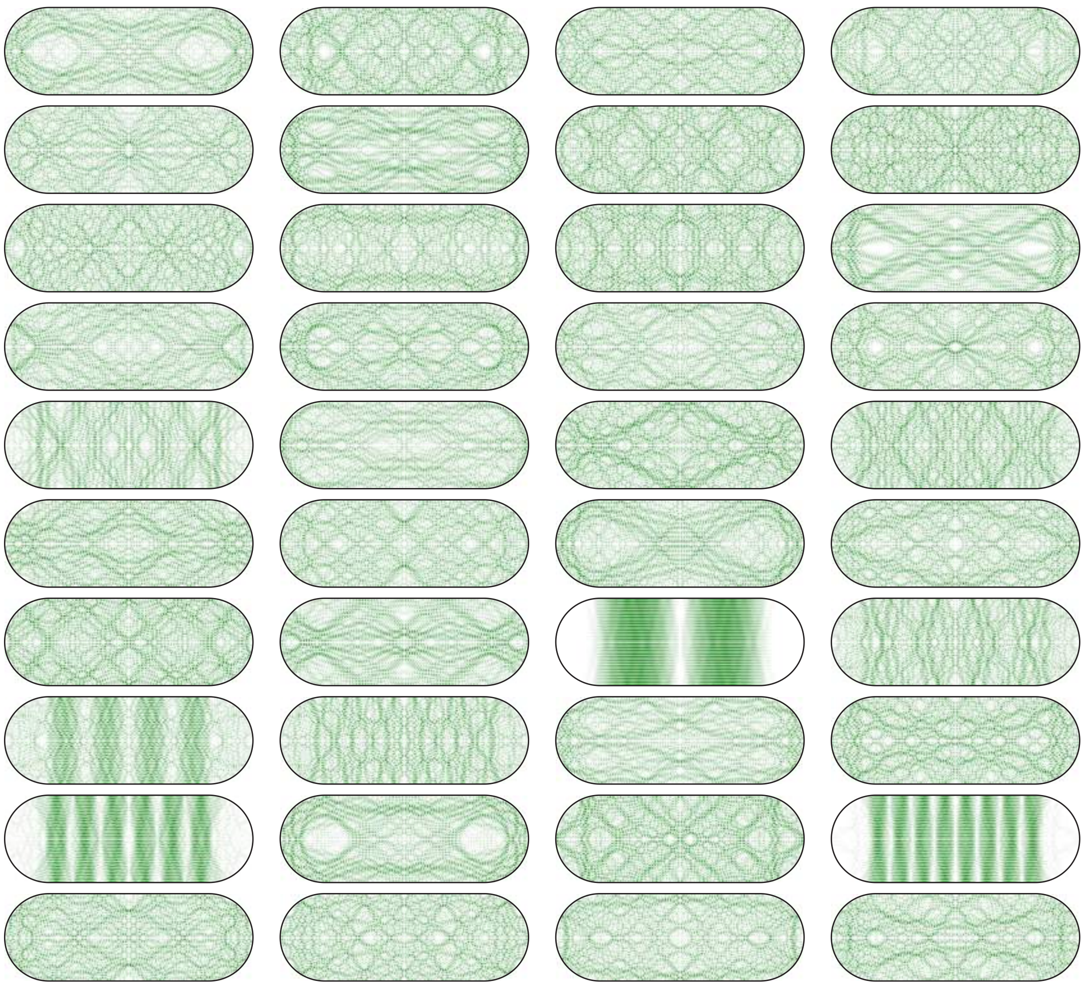
If the geodesic flow is ergodic, this implies

$$Var_\lambda(a(x, D_x)) \underset{\lambda \longrightarrow +\infty}{\longrightarrow} 0$$

if a has zero mean.



QUE conjecture :

Conjecture (Rudnick, Sarnak 94)

On a negatively curved manifold, we have convergence of the whole sequence : $\langle \psi_k, a(x, D_x)\psi_k \rangle_{L^2(M)} \longrightarrow \int_{|\xi|=1} a(x,\xi) dx d\xi$.

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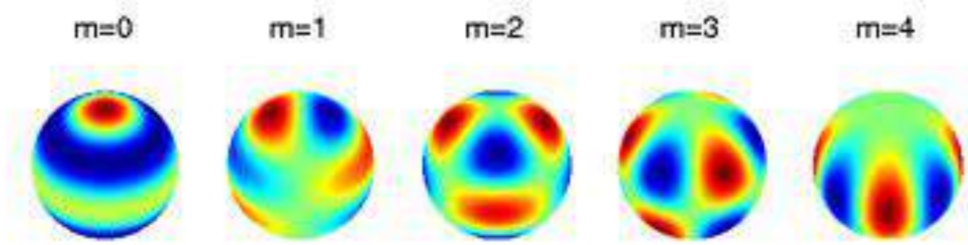
Proven by E. Lindenstrauss in the special case of arithmetic congruence surfaces, for joint eigenfunctions of the Laplacian and the Hecke operators.

A-Nonnenmacher (06) proved a weaker statement valid in greater generality.
Let M have negative curvature. Assume

$$\langle \psi_k, a(x, D_x)\psi_k \rangle_{L^2(M)} \longrightarrow \int_{|\xi|=1} a(x,\xi) d\mu(x,\xi)$$

Then μ must have positive Kolmogorov-Sinai entropy.

The sphere.



Flat tori

(Jakobson-Bourgain 97, Jaffard 90, A-Macià 2012)
It's not possible for a sequence of eigenfunctions to concentrate on a closed geodesic.

On each cylinder of periodic orbits, the limit measure must be absolutely continuous.

QE on discrete graphs

Since the 90s there has been the idea of using graphs as a testing ground/toy model for quantum chaos.

Smilansky, Kottos, Alon,...
Keating, Berkolaiko, Winn, Piotet, Marklof...

Here we focus on the case of large regular (discrete) graphs.
Let $G_N = (V_N, E_N)$ be a $(q+1)$-regular graph of size N ($V_N = \{1, \ldots, N\}$).

We look at the limit $N \longrightarrow +\infty$.

We assume that G_N has "few" short loops ($=$ converges to a tree in the sense of Benjamini-Schramm).

Theorem

(A-Le Masson, 2013) Assume that G_N has "few" short loops and that it forms an expander family.
Let $(\phi_i^{(N)})_{i=1}^N$ be an ONB of eigenfunctions of the laplacian on G_N.
Let $a = a_N : V_N \longrightarrow \mathbb{C}$ be such that $|a(x)| \leq 1$ for all $x \in V_N$.
Then

$$\lim_{N \longrightarrow +\infty} \frac{1}{N} \sum_{i=1}^{N} \left| \sum_{x \in V_N} a(x) |\phi_i^{(N)}(x)|^2 - \bar{a} \right|^2 = 0.$$

- Also works on shrinking spectral intervals
- Applies to random regular graphs. In that case there also exists a probabilistic proof (Geisinger 2013) in the case where $a(x)$ is chosen independently of G_N.

More general version

Theorem

(A-Le Masson, 2013) Assume that G_N has "few" short loops and that it forms an expander family.
Let $(\phi_i^{(N)})_{i=1}^N$ be an ONB of eigenfunctions of the laplacian on G_N.
Let $K_N : V_N \times V_N \longrightarrow \mathbb{C}$ be a matrix such that $d(x,y) > D \Longrightarrow K_N(x,y) = 0$. Assume $|K_N(x,y)| \leq 1$. Then

$$\lim_{N \longrightarrow +\infty} \frac{1}{N} \sum_{i=1}^{N} \left| \langle \phi_i^{(N)}, K_N \phi_i^{(N)} \rangle - \overline{K_N}(\lambda_i) \right|^2 = 0.$$

$$\overline{K_N}(\lambda_i) = \sum_{x,y} K(x,y) \Phi_{sph,\lambda_i}(d(x,y)).$$

Theorem

(Brooks-Lindenstrauss 2011) Assume that G_N has "few" loops of length $\leq c \log N$.
For $\epsilon > 0$, there exists $\delta > 0$ s.t. for every eigenfunction ϕ,

$$B \subset V_N, \sum_{x \in B} |\phi(x)|^2 \geq \epsilon \Longrightarrow |B| \geq N^{\delta}.$$

Proof also yields that $\|\phi\|_\infty \leq |\log N|^{-1/4}$.

Sketch of proof : 1) work with non-backtracking RW instead of simple RW

$G = (V, E)$ graph, $|V| = N$, $\mathcal{A} : \ell^2(V) \longrightarrow \ell^2(V)$ (self-adjoint) defined by

$$\mathcal{A}f(x) = \sum_{y \sim x} f(y).$$

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Define $B =$ set or oriented edges of G, and $\mathcal{B} : \ell^2(B) \longrightarrow \ell^2(B)$ by

$$\mathcal{B}f(e) = \sum_{o(e')=t(e), e' \neq \hat{e}} f(e').$$

Note that $\mathcal{B}$ is not self-adjoint but $\mathcal{B} = I\mathcal{B}^* I$ where I is the edge-reversal involution

$$If(e) = f(\hat{e}).$$

For regular graphs, the spectrum and eigenfunctions of $\mathcal{B}$ are explicit in terms of those of $\mathcal{A}$.

- each eigenvalue $\lambda = 2\sqrt{q}\cos(s \ln q)$ ($s \in \mathbb{R} \cup i\mathbb{R}$) of $\mathcal{A}$ gives rise to two eigenvalues $q^{1/2 \pm is}$ of $\mathcal{B}$.

- the $N(q-1)$ other eigenvalues are ± 1, each with multiplicity $\frac{N(q-1)}{2}$ (rank of fundamental group of G -1).

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  The eigenfunction ϕ_s for $\mathcal{A}$ gives rise to the two eigenfunctions of $\mathcal{B}$,

$$f_s^{\pm}(e) = \phi(t(e)) - \frac{1}{q^{1/2 \pm is}} \phi(o(e)).$$

- the $N(q-1)$ other eigenvalues are ± 1, each with multiplicity $\frac{N(q-1)}{2}$ (rank of fundamental group of G -1).

2) Definition of the quantum variance for the non-backtracking operator

Consider $K(e, e') : B \times B \longrightarrow \mathbb{C}$ with the property that

$$K(e, e') \neq 0 \Longrightarrow \exists k \leq D, \mathcal{B}^k(e, e') \neq 0.$$

(K may be seen as a function on the set of geodesic segments of length $\leq D$).

Define

$$Var(K) = \frac{1}{N} \sum_{j=1}^{N} |\langle If_{s_j}^{-}, Kf_{s_j}^{+} \rangle|^2.$$

This is built so that

$$Var([\mathcal{B}, K]) = 0$$

for all K.

3) Dynamical interpretation

Notice that

$$[\mathcal{B}, K] = dK$$

("derivative along geodesic flow") and that

$$Var(K) \leq C\frac{1}{N}\sum_{e,e'}|K(e,e')|^2$$

if $D \leq \ girth$.

Uniform mixing of $\mathcal{B}$ on a family of graphs (=expanding property) then implies that

$$Var(K_N) \underset{N\longrightarrow\infty}{\longrightarrow} 0$$

if $\mathrm{Tr}(K_N\mathcal{B}^k) = 0$ for all k.

Perspectives

This method seems adaptable to

- quotients (with large girth) of $(F_d, \langle a_1, \ldots, a_d \rangle)$ with weights $p(x, xa_i) = p(a_i)$ symmetric.
- $\Delta + v$, with v deterministic, having some kind of periodicity ; probably also $v(x)$ random iid (Anderson model).
- some non-regular graphs ? ?

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For a lot of graphs (not only regular ones) there is an explicit way to transform solutions of

$$(\Delta + v)\phi = \lambda\phi$$

in $\ell^2(V)$ to solutions of $\mathcal{B}f = \alpha_\lambda f$ in $\ell^2(B)$, where α_λ is a function on B.

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When you have a family of graphs and you need to control the behaviour of the functions α_λ...