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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Justin Hilburn **Email/Phone:** jhilburn@uoregon.edu

Speaker's Name: Konstantin Ardakov

Talk Title: Equivariant D-cap modules on rigid analytic spaces

Date: 11 / 19 / 2014 **Time:** 11 : 00 am / pm (circle one)

List 6-12 key words for the talk: Lie Algebroid, Rigid Space, Frechet Stein, Coadmissible module, D-cap Module

Please summarize the lecture in 5 or fewer sentences:
Ardakov discussed an analogue of the Beilinson-Bernstein Localization theorem coadmissible D(G,K)-modules with G-equivariant D-modules on the the flag variety of a p-adic group. To do this he had to explain a generalization of the theory of Lie algebroids to rigid spaces.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☑ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☑ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
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- ☑ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

<u>Motivation</u> understand admissible local reps of p-adic groups - Geometrically study

$$\left\{ \text{coadmissible } D(G,K)\text{-modules} \right\}$$

- $\widehat{U(\mathfrak{g}_K)} \subset D(G,K)$ the Arens-Michael envelope
- $\exists$ good theory of coadmissible $\widehat{\mathcal{D}}$-modules on rigid analytic spaces + S. Wadsley.

<u>Expected Theorem</u>
$$\left\{ \begin{matrix} \text{coadmissible} \\ D(G,K) \underset{Z(\mathfrak{g}_K)}{\otimes} K \end{matrix} \right\} \cong \left\{ \begin{matrix} G\text{-equivariant} \\ \widehat{\mathcal{D}}\text{-modules} \\ \text{on } (\mathbb{G}/\mathbb{B})^{an} \end{matrix} \right\}$$

At least when $G \leq \mathbb{G}(L)$ $\quad$ L/K finite

G: compact $\quad$ $\mathbb{G}$: Split semisimple

$$\left\{ \begin{matrix} \text{coadmissible} \\ \widehat{\mathcal{D}} \rtimes G\text{-modules} \\ \text{on } X/G \end{matrix} \right\}$$

X adic space

X/G topological quotient of topological underlying space of X

To define this need either adic spaces or Raynaud's approach via formal models.

§ Lie algebroids

R fixed base ring, X/R scheme

~~A Lie algebroid consists of a quasicoherent~~
~~$(L, \sigma, [\,,\,])$ other~~

A Lie algebroid is a triple $(L, \sigma, [\,,\,])$ where

- L is a q coh $\mathcal{O}_X$-module
- $\sigma : L \to T_X$ $\mathcal{O}_X$-linear
- $[\,,\,] : L \times L \to L$ R-Lie bracket

$$\left.\begin{array}{l} \bullet\ [\sigma v, \sigma w] = \sigma[v,w] \\ \bullet\ [v, aw] = a[v,w] + \sigma(v)(a)w \end{array}\right\} \begin{array}{l} \forall\, v,w \in L \\ \forall\, a \in \mathcal{O}_X \end{array}$$

Ex
1) $L = T_X$, $\sigma = 1$.
2) T is algebraic group over R and $\xi : \tilde{X} \to X$ T-torsor, then $(\xi_* T_{\tilde{X}})^T$ is a Lie algebroid.
3) $c \in R$, cL is a Lie algebroid when L is.

PBW theorem

Can form enveloping algebra $U(L)$ generated by $\mathcal{O}_X$ and L

Thm (Rinehart) $\exists\ S(L) \twoheadrightarrow \mathrm{gr}\, U(L)$ which is an isomorphism when L is locally free.

§ Formal schemes

R - cdvr, $\pi \in R$ uniformizer, $k = R/\pi R$, $K = R[\frac{1}{\pi}]$

formal scheme = q.-c. formal sch / Spf R
flat over Spf R
and locally Spf $R\langle u_1, \ldots, u_m \rangle / I$

If X is a formal scheme $\mathcal{T}_X = \mathrm{Der}_R^{\mathrm{cont}} \mathcal{O}_X$
can define coherent Lie algebroids similarly.

Def $\widehat{U(\mathcal{L})} = \varprojlim U(\mathcal{L})/(\pi^a)$
$\widehat{U(\mathcal{L})}_K = \widehat{U(\mathcal{L})} \otimes_R K$

Fact If X is smooth / Spf R
then $\widehat{U(\mathcal{T}_X)}_K \cong \widehat{\mathcal{D}}^{[0]}_{X,\mathbb{Q}}$ Berthelot

Thm $\mathbb{G}$-split connected s.c. reductive / R
$\mathbb{B} \subseteq \mathbb{G}$ [illegible] Borel. $X = \mathbb{G}/\mathbb{B}$. Then
for all $c \in R \setminus \{0\}$:

1) $\Gamma : \left\{ \begin{array}{c} \text{coherent} \\ \widehat{U(c\mathcal{T})}_K\text{-mods} \end{array} \right\} \xrightarrow{\cong} \left\{ \text{f.g. } \Gamma(X, \widehat{U(c\mathcal{T})}_K)\text{-mods} \right\}$

2) $\Gamma(X, \widehat{U(c\mathcal{T})}_K) = \widehat{U(c\mathfrak{g})}_K \otimes_{Z(\mathfrak{g})_K} K$

§ Passing to $c = 0$

Def Let A be a K-algebra. It is Fréchet-Stein
if $\exists$ tower $\cdots \to A_2 \to A_1 \to A_0$ of Noetherian
Banach algebras / K st $A = \varprojlim A_n$

Example a) $A_n = U(\pi^n \mathfrak{g})_K$

$\Rightarrow \varprojlim A_n =: \widehat{U}(\mathfrak{g}_K)$

b) $\mathfrak{g} = K \Rightarrow A_n = K\langle \pi^n x \rangle$

& $\widehat{U(\mathfrak{g}_K)} = \cap K\langle \pi^n x\rangle = K\{x\}$

$= \mathcal{O}(A^{1, an}_{rig})$

Schmidt considered $\varprojlim \widehat{U(\pi^n \mathfrak{g})_K}$ on $\widehat{\mathbb{G}/\mathbb{B}}$.

Digression

Defn A Fréchet–Stein

An A-module is coadmissible if $\forall n > 0$

$A_n \otimes_A M$ is f.g.

Thm $\mathcal{C}A = \{\text{coadmissible } A\text{-modules}\}$ is abelian.

Problem Zariski topology on $\widehat{\mathbb{G}/\mathbb{B}}$ is too coarse !!

Ex Spf $R\langle x\rangle$ $\leftarrow$ $K\langle x, \partial\rangle$

Spf $R\langle x/\pi\rangle$? $K\langle \frac{x}{\pi}, \pi\partial\rangle$

Pullback of Lie algebroids

Def Let L be a Lie algebroid on a scheme X. $\varphi: Y \to X$. If $\varphi(V) \subseteq U$ affine

$\varphi^* L(V) = \mathcal{O}(V) \otimes_{\mathcal{O}(U)} L(U) \underset{\mathrm{Der}_R(\mathcal{O}(U), \mathcal{O}(V))}{\times} \mathrm{Der}(\mathcal{O}(V))$.

with Lie bracket

~~[illegible]~~

$$[(b\otimes \ell, \delta), (b'\otimes \ell', \delta')]$$
$$= \left(bb' \otimes [\ell,\ell'] + b\,\delta(b')\otimes \ell' - b'\delta'(b)\otimes \ell, [\delta,\delta']\right)$$

Fact $\varphi^! L \to \varphi^* L$ is an iso

$\Longleftrightarrow$

1) $\mathrm{Der}_{\mathcal{O}(U)}\mathcal{O}(V) = 0$

2) $\forall\, v \in L(V)$ $\exists\, \tilde{\sigma}(v) \in \mathrm{Der}_K(V)$

st $\varphi \circ \sigma(v) = \tilde{\sigma}(v) \circ \varphi$.

Can happen for more general morphism than étale ones.

§ Back to formal schemes.

Def Let L be a coherent Lie algebroid on a formal scheme X.

A morphism $\varphi: Y \to X$ is L-étale if

1) φ is rig-unramified

2) $\varphi^! L \to \varphi^* L$ is an iso.

continuous differentials

$\Omega^1_{Y/X} \otimes_R K = 0$

Ex $X = \mathrm{Spf}\ R\langle x\rangle$

$L = R\langle x\rangle\, \pi^n \partial_x$

$Y = \mathrm{Spf}\ R\langle x/\pi^m\rangle$

Then $\pi^n \partial_x = \pi^{n-m}\, \partial_{x/\pi^m}$ shows φ is L-étale iff $n \geq m$.

Prop Let $X(L) := \{ e: Y \to X \mid e \text{ is } L\text{-etale} \} \subseteq$ full rig-etale $/X$

This category has finite products and we have a presheaf
$$\widehat{\mathcal{U}(L)}_K (Y) = \Gamma(Y, \widehat{\mathcal{U}(e^*L)}_K)$$
on $X(L)$.

Thm Suppose L is coherent and locally free then $\widehat{\mathcal{U}(L)}_K$ satisfies the sheaf condition for

a) Zariski covering

b) admissible formal blowups

If X is rigid affine, higher cohomology vanishes.

§ Rigid spaces

Lemma Let $f: Y \to X$ be rig-etale. Then $\exists\ n \geq 0$, st. f is $\pi^n L$-etale $\forall n \geq 4$.

Def $\widehat{\mathcal{U}(L_K)} := \varprojlim \Gamma(Y, \widehat{\mathcal{U}(f^*(\pi^m L))}_K)$.

Thm 2 Suppose L_K is rig coherent + locally free. Then

a) $\widehat{\mathcal{U}(L_K)}$ is a rig-etale sheaf

b) If Y is rig-affine & then

$\Gamma(Y, \widehat{U(L_K)})$ are Frechet-Stein.

Let $\mathcal{C}_Y = \{$ coadmissible $\Gamma(Y, \widehat{U(L_K)})$-modules $\}$

Thm 3 a) If $Z \to Y$ any open embedding between aff-affines Y & Z, $\exists$ an exact localization functor
$M \mapsto \widehat{U(L_K)} \underset{\widehat{U(L_K)}(Y)}{\otimes} M$

b) Kiehl's thm holds.