**Around the Canonical Base Property**

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The canonical base property was a property used by Pillay and Ziegler to bypass the use of Zariski geometries in the proofs of Manin-Mumford type results. It was first proved for fields such as differentially closed fields.

We assume that we have a good notion of dimension, independence, and generics. Let $S \subseteq X \times Y$; we look at $S$ as a uniformly varying family $S_x = \{y \in Y \mid (x,y) \in S\}$. If $x \neq x'$ then $S_x$ and $S_{x'}$ should not have the same generics. Take $a \in X$, $b$ generic in $S_a$ and consider $S^b = \{x \mid (x,b) \in S\}$. The canonical base property gives very strong conditions on $S^b$.

In the case of complex manifolds, it says that it's Moishezon (algebraic). In the case of differentially closed field (models of $DCF_0$), it tells us that this set $S^b$ is internal to the constants, i.e. birational with a variety with points in the constant field. Similar constraints hold in the case of difference fields.

**Model Theoretic Setting**

Let $T = T^{eq}$ be supersimple, and consider types of finite rank. We work in some monster model $\mathcal{U} \models T$.

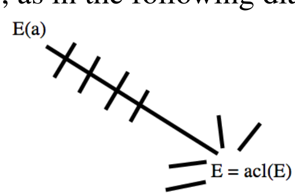
For example, take $T = DCF_0$, which is existentially closed in the class of differential fields with $\mathcal{U}$ a large saturated model of $T$. For finite rank, consider $E = \mathrm{acl}(E)$ an algebraically closed field differential field. Take some tuple $a \in \mathcal{U}$ and assume that $E(a)$ is closed under the derivation. Then **finite rank** means that $trdeg(E(a)/E) < \omega$ and that $E(a)$ is a differential field. Now let $E(a)/E$ be finite rank and consider $E(b)$ for $b$ a generic, and let $E' \subseteq E(a)$ with $\mathrm{acl}(E') \subset \mathrm{acl}(Ea)$ strictly. Then $b \not\perp_{E'} a$. $Ea = \mathrm{CB}(b/Ea)$. This gives strong conditions on $\mathrm{tp}(a/Eb)$.

The CBP will tell you that $\mathrm{tp}(a/Eb)$ is almost internal to a non-one-based type of rank 1, i.e. $F \perp_{Eb} a$ with $F \supseteq Eb$ with $\mathrm{acl}(Fa) = \mathrm{acl}(Fe)$ where $e$ is a tuple of realization of types of rank 1 which is not 1-based. In the case of $DCF_0$, the tells us that if $S^b$ is the differential locus of $a$ over $E < b >$ then (almost; morally) $S^b \cong W(C)$ for some variety $W$ over the constant field $C$.

**Semi-minimal analysis** Let $E = \mathrm{acl}(E)$, and consider some extension $Ea := \mathrm{dcl}(E \cup \{a\})$. Then consider a set of elements $a_1, \cdots, a_n \in \mathrm{dcl}(Ea)$ and a sequence $E \subseteq E(a_1) \subseteq E(a_2) \subseteq \cdots \subseteq E(a_n) = E(a)$ so that if $E(a_i) \subseteq E' \subseteq E(a_{i+1})$ then either $E' \subseteq \mathrm{acl}(Ea_i)$ or $a_{i+1} \in \mathrm{acl}(E')$. Such a sequence exists if you have a good dimension theory.

Zilber Principle: Each type $\mathrm{tp}(a_{i+1}/E(a_i))$ is either 1-based or (almost) internal to a non-1-based type of rank 1.

CBP (informally): Whenever $Ea = \mathrm{CB}(b/Ea)$ then the fibration that gives us an analysis splits, as in the following diagram




A definable set $D$ (defined over $E$) is 1-based if whenever $a_1, \cdots, a_n \in D$, $F \supset E$, then $\mathrm{acl}(Ea_1, \cdots, a_n) \cap \mathrm{acl}(F) = C$ then $a_i \cdots a_n \perp_C F$. This never happens in fields: consider $a, b, c$ transcendental and independent over $\mathbb{Q}$ and let $d = ac + b$. Then $\mathbb{Q}(a,b)^{alg} \cap \mathbb{Q}(c,d)^{alg} = \mathbb{Q}^{alg}$ but they are clearly not independent. This is not 1 based!

In general there is a (false) conjecture/principle that says that if you are not one-based, it is because of the presence of a field. This is false in general, but true for lots of fields with extra structure.

Consider the theory of vector spaces over a field $k$ in the language $\mathcal{L} = \{+, -, 0, \{c\cdot\}_{c\in k}\}$ and work in a monster $\mathcal{U}$. Consider a tuple $a_1 \cdots, a_n$ and let $F =< F >$. Then consider $\mathrm{CB}(a_1 \cdots a_n/F)$ when $< a_1, \cdots, a_n > \cap < F >= C$, then the type of the $a_i$ over $F$ is 1-based by simple facts about linear independence.

What happens if you know what the 1-based types are? Does every supersimple theory have the CBP? No, there are even $\omega$-stable counterexamples (!).

How do you show that the CBP holds? We consider $p$-analyzability, wherein *all* types in the analysis are almost internal to the same $p$. One can show that in the case of $E$, $a$, and $b$ as above (i.e. $Ea = \mathrm{CB}(\mathrm{tp}(b/Ea))$) that $\mathrm{acl}(Ea) = \mathrm{acl}(Eb_1 \cdots b_n)$ such that each $\mathrm{tp}(b_i/E)$ is $p_i$-analyzable with $p_i$ not 1-based and of rank 1. Hence, to show that the CBP holds it suffices to consider this case. By an analysis of such types, you can show the CBP for existentially closed difference fields(ACFA) by running through the Pillay-Ziegler proof.

Open problem: For $SCF_{1,p}$. We know what the non-1-based types look like, but we don't know about the analysis of all types.

One of the consequences of the CBP: a descent result. Let $p$ be a non-1-based type of rank one. Consider $E \subseteq B_1, B_2$, all algebraically closed such that $B_1 \cap B_2 = E$, $\mathrm{tp}(B_2/E)$ is almost $p$-internal, and let $a_1$ be over $B_1$ and $a_2$ over $B_2$ such that $a_1 \mathop{\smile\!\!\!\!\!\!\mid}_{B_1} B_2$ and $a_2 \mathop{\smile\!\!\!\!\!\!\mid}_{B_2} B_1$ with $a_2 \in \mathrm{acl}(B_1 B_2 a_1)$. Then there is $d \in \mathrm{acl}(B_2 a_2)$ with $d \mathop{\smile\!\!\!\!\!\!\mid}_{E} B_2$ and $\mathrm{tp}(a_2/Ed)$ is almost $p$-internal.

Once translated into the language of (differential) algebraic variety with $V_1$ the locus of $a_1/B_1$, $V_2$ the locus of $a_2/B_2$, then $V_{2,B_1B_2}$ has a quotient $V_{0,B_1B_2}$ over E whose generic fiber is $p$-internal.

For difference varieties: Let $V_1$ be a variety, $B_1 = K \vDash ACF$, $B_2 = K(t) = L$ transcendental over $K$, and we have a system $(V_1, \phi)$ with $\phi$ a rational dominant self map and $(V_2, \psi)$ similar defined over $L$. Then we have a rational map $g : V_1 \to V_2$ making the diagram

$$\begin{array}{ccc} V_1 & \overset{g}{\dashrightarrow} & V_2 \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\psi} \\ V_1 & \longrightarrow & V_2 \end{array}$$

commute. Then $(V_2, \psi)$ has a quotient $(V_0, \psi_0)$ defined over $K$ with $\deg(\psi_0) = \deg(\psi)$. In particular if $\dim V_1 = 1$ and $\deg(\psi) > 1$ then $V_1$ had to be a finite cover of $V_0$.