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Mathematical Sciences Research Institute

17 Gauss Way Berkeley, CA 94720-5070 p: 510.642.0143 f: 510.642.8609 www.msri.org

NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: James Freitag Email/Phone: freitag@math.berkeley.edu

Speaker's Name: Zoé Chatzidakis

Talk Title: An application of difference fields to algebraic dynamics

Date: 05/12/14 Time: 2:00 am / pm (circle one)

List 6-12 key words for the talk: difference fields, algebraic dynamics,

Please summarize the lecture in 5 or fewer sentences: This talk explains an application of the model theory of difference fields to algebraic dynamics. Specifically, the talk concentrates on generalizing a descent theorem of Baker from the projective line to arbitrary varieties.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☐ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☐ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - Computer Presentations: Obtain a copy of their presentation
  - Overhead: Obtain a copy or use the originals and scan them
  - Blackboard: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
  - Handouts: Obtain copies of and scan all handouts
- ☐ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☐ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- ☐ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

Motivation:

Let $K = k(t)$. In this talk, we will be interested in K-rational points of various objects.

Let $V \subseteq \mathbb{P}^n$ be a variety $/K$.

Let $\phi : V \longrightarrow V$ be a dominant rational map. (again $/K$).

If $p \in V(K)$ with $p = (x_0 : \dots : x_n)$ with $x_i \in k[t]$ relatively prime, then

$$h(p) := \max \{\deg x_i(t)\}.$$

Fix $N \in \mathbb{N}$

$$S := \{P \in V(K) \mid h(P) \leq N\}$$

This set is constructible over k, and we will denote it by M_N from now on (as in Antoine's talk).

Assume, for all m, $\{P \in V(K) \mid P, \phi(P), \dots, \phi^{(m)}(P) \in M_N\}$ is Zariski dense in V.

This "should imply" that $(V, \phi) \cong (U, \psi)$, where (U, ψ) is defined over k.

Matt Baker: If $V=\mathbb{P}^1$ and $\deg\phi \geq 2$ then the result is true.
(The result that "should be true" from the previous page)

Difference Fields:

$\boxed{L, \sigma \in \mathrm{End}(L)}$, K, V, ϕ as above with V abs. irred.

This is what is meant by a "difference field".

For us, the pertinent difference field is

$$\begin{array}{ccc} K(V) & \text{with} & \phi^* \\ \parallel & & \parallel \\ L & & \sigma \end{array}$$

e.g. $V = \mathbb{A}^1$

$\phi : x \longmapsto x+1$

$K(\mathbb{A}^1) = K(x)$

$\phi^*(f(x)) = f(x+1)$.

Keep in mind, in this talk, everything is up to birationality.

<u>Thm:</u> (C.-Hrushovski)

K, k, ϕ, V as above.

Then there is some $(\mathcal{U}, \theta)/k$ and a dominant map $g : (V, \phi) \longrightarrow (\mathcal{U}, \theta)$ s.t. $\deg(\phi) = \deg(\theta)$ (and the generic fiber is $\underbrace{\mathrm{Fix}(\sigma) \text{ internal}}$)

$$\begin{tikzcd} V \arrow[r, "g"] \arrow[d, "\phi"'] \arrow[dr, phantom, "\circlearrowright"] & \mathcal{U} \arrow[d, "\theta"] \\ V \arrow[r, "g"] & \mathcal{U} \end{tikzcd}$$

Take $a \in V$ generic.
$(g^{-1}(g(a)), \phi\restriction_{g^{-1}(g(a))})$ is $\mathrm{Fix}(\sigma)$-internal.

This result says <u>nothing</u> when (V, ϕ) is $\mathrm{Fix}(\sigma)$-internal:

embedding of diff. fields

$$\underbrace{(K(V), \phi^*)}_{\text{A difference field}} \longhookrightarrow (\mathcal{U}, \sigma)$$

where $(\mathcal{U}, \sigma)$ is an existentially closed difference field.

$$\begin{tikzcd} V \arrow[dr, "h"] \arrow[dd, "\phi"'] & \\ & W \\ V \arrow[ur, "h^\sigma"'] & \end{tikzcd}$$

$\exists$ rational map $h : V \to W$

W is def. $/\mathrm{Fix}(\sigma)$.

(Notice if h is def. $/\mathrm{Fix}(\sigma)$, this is very strong.)

What are the $\mathrm{Fix}(\sigma)$-internal varieties?

e.g. (K, σ) a diff. field. (Picard-Vessiot)

$A \in GL_m(K)$

$\sigma(X) = AX$ for $X \in GL_m(\mathcal{U})$. or just a vector even.

e.g. (Translation variety)

$G \curvearrowright Y$ over $\mathrm{Fix}(\sigma)$.

Let $g \in G(\mathcal{U})$, $x \in Y$, $\sigma(x) = gx$ is a translation variety.

If $Y = G$, all solutions are obtained from one solution by multiplying on the right by $G(F)$.

(similar but more complicated results without assumption $G = Y$.)

We want to show that in many cases $\mathrm{Fix}(\sigma)$-internal diff. varieties are related to translation varieties.

Let Q/K be a σ-variety:

$Q = \{x \in V / \sigma(x) = \phi(x)\}$ for some birational $\phi : V \longrightarrow V^{\sigma}$, all def. $/K$.

After some (Z. says this part is not hard) work,

$$K(Q(\mathcal{U})) = K(a)\,Fix(\sigma)$$

This is some "fundamental solution".

<u>Thm</u>: Let K, Q be as above, with Q F-internal. Assume if $a \in Q$ generic, $K(a) \cap F = K \cap F$. Then $Q \cong$ translation variety in the following cases:

1) $K \models ACF$, $Fix(\sigma) \cap K$ is pseudofinite.

2) $K \models ACF$, $K \subseteq F$. In the latter case the group G is abelian.

$Q \cong \{x \in A / \sigma(x) = x + a\}$ where

A is an abelian group.

Sketch of the proof:

By the height hypothesis, one gets

$r>0$, $(W, \cancel{\phi})$ def. $/k$ and ψ

$h:(W,\psi) \longrightarrow (V,\phi^r)$. dominant.

Going to $\overline{k}$ and $\overline{K}$,

$(W,\psi) \cong (B, t_b) \twoheadrightarrow (V,\phi) = (A, t_a)$

We may assume A is simple.

Case 1: $A = \mathbb{G}_a$ $\quad \sigma(x) = x + a$

If $a = 0$ then ✓.

If $a \neq 0$ $\quad \sigma\left(\frac{x}{a}\right) = \frac{x}{a} + 1$

which is defn $/k$.

Case 2: A is semiabelian ← very few endomorphisms

$\Downarrow$

B is semiabelian

$$\begin{tikzcd}
(B, t_b) \arrow[rr] \arrow[d, "f_2"'] & & (A, t_a) \arrow[dll, "f_1"] \\
(A', t_a') \arrow[ur, phantom, "\circlearrowleft"] & &
\end{tikzcd}$$

– {with f_1 an isogeny.

– {A' def$/k$.

In response to a question about fields of definition:

Given:

$$(V_1, \phi_1)_{/K_1} \xrightarrow{g} (V_2, \phi_2)_{/K_2}$$

then you know g can be assumed to be defined over $(K_1 K_2)^{alg}$.

This follows by stable embeddedness of alg. cl. fields – this is only a statement about algebraic geometry.