Conformal Invariance of Ising Model Interfaces

Clément Hongler

MSRI workshop on conformal invariance and statistical mechanics

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Notes taken by Samuel S Watson

Recall the definition of the Ising model

$$\mathbb{P}[\sigma] = \frac{1}{Z} e^{-\beta H(\sigma)},$$

where $H(\sigma) = \sum \sigma_i \sigma_j$ over edges in the graph. Thus the model favors local alignment. We consider the model on $\Omega \cap \delta\mathbb{Z}^2$, and we might choose to impose some boundary conditions, specifying spins along the boundary. As $\delta \to 0$, there is a phase transition at $\beta_c = \frac{1}{2}\log(\sqrt{2}+1)$.

It turns out that if we choose Dobrushin boundary conditions ($+$ on one arc, $-$ on the complement), the interface is converges to SLE_3. If we consider three arcs with $+$, $-$, and free boundary conditions, we get something called dipolar SLE. The full collection of interfaces converges to CLE_3.

The dipolar interface is defined by starting at the vertex where the $\pm$ arcs meet and separating $+$ faces from $-$ faces. For definiteness, we turn left whenever there is an ambiguity.

Theorem 1. The dipolar interface converges in law as $\delta \to 0$ to dipolar SLE_3.

Recall Loewner's theorem in the strip, which gives a way to encode a path in the strip in terms of a real-valued process, by way of the ODE

$$\partial_t g_t(z) = \coth\left(\frac{1}{2}(g_t(z) - U_t)\right)$$

We can conformally map our domain to the strip with the three points separating boundary arcs mapped to $\pm\infty$ and 0.

Idea of proof of convergence theorem. We first establish precompactness, which guarantees subsequential scaling limits. The second, more important step, involves identifying the limit. We use the rather common *martingale* technique. The philosophy here is that martingales give stochastically conserved quantities, which in turn correspond to symmetries of probability measures. □

Suppose we have a subsequential limit γ, and define the conformal map $\varphi_t : \Omega \setminus \gamma[0,t] \to \mathbb{S}$. Key claim

$$\left(|\phi_t'(z)|^{1/2} \coth\left(\frac{1}{2}\varphi_t(z)\right)\right)_{t\geq 0}$$

is a local martingale. Considering the driving function corresponding to γ, and differentiate the key claim with respect to z. We find that V_t and $(V_t^2 - 3t)_{t\geq 0}$ are local martingales.

How does one prove the key claim? We find a quantity on the discret level that behaves like a discrete-time martingale as the dipolar interface grows and show that it converges to the martingale in the limit statement. Let $Z(\omega_\delta, a, b, c)$ be the partition function of the model, and let

$\tilde{Z}(\Omega_\delta, z, a, b, c)$ be the partition function of the model with boundary values modified so that the arc between b and z is negative. The ratio $\tilde{Z}/Z$ is our discrete-time martingale. In fact, it is easy to show that this is a martingale, on a combinatorial level, just by considering one step of the exploration process.

The remaining step is to show that this martingale converges to the function specified in the key claim. We use Kramers-Wannier duality. We rewrite the ratio of partition functions as a ratio of expectations in a dual model

$$\frac{\mathbb{E}^*_{\Omega_\delta}[\sigma_a \sigma_z]}{\mathbb{E}_{\Omega^*_\delta}[\sigma_a]}.$$

The idea for this is to rewrite the partition function as a sum involving the loops in a dual model. Now once we have this representation, we prove convergence by proving convergence separately for the numerator and denominator. However, we face the difficulty that spin boundary correlations are not as easy to calculate.

We make use of the FK representation of the Ising model. The FK model is a dependent percolation model, and it represents geometrically the influence between the spins. The model is defined by

$$\mathbb{P}(\omega) = Z^{-1}(p/(1-p))^{\#\omega} q^{\#\text{clusters}(\omega)}.$$

The Ising model is obtained from the FK model by assigning i.i.d. $\pm$ spins to each cluster in the FK model. Then

$$\mathbb{E}_{\Omega^*_\delta}[\sigma_a \sigma_z] = \mathbb{P}_{\Omega^*_\delta}[a \leftrightarrow z] \qquad \mathbb{E}_{\Omega^*_\delta}[\sigma_a] = \mathbb{P}_{\Omega^*_\delta}[a \leftrightarrow (b, c)]$$

We can use the domain Markov property of the model and standard results for the FK model to obtain these convergence results.

Remark: the nice thing about the Ising model is that when something seems plausible, it's usual possible to prove it.

Extension: We can extend this dipolar SLE path until it hits some fixed point d in the free boundary. (The idea is to pretend there are boundary conditions forcing the path toward d, but only for the purpose of defining the path, not the model itself). It turns out that this path converges to SLE$_\kappa\left(\frac{\kappa-6}{2}, \frac{\kappa-6}{2}\right)$. The main idea is to identify the excursions of this process, and control what happens on $[bc]$, and show that the curve bounces nicely. This uses new RSW estimates in terms of extremal length, plus some basic facts about Bessel processes.