Quantum Riemann Surfaces and Box-counting

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MSRI workshop on conformal invariance and statistical mechanics

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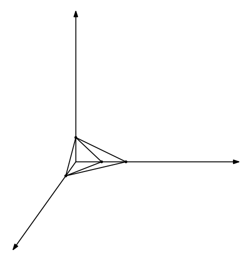
Notes taken by Samuel S Watson

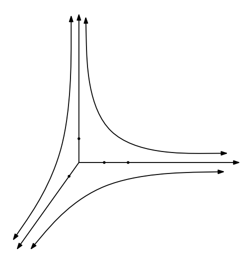
Consider a (noncompact) convex polyhedron Δ in $\mathbb{R}^3$, such as an octant. Consider $\Delta_{\mathbb{Z}} := \Delta \cap \mathbb{Z}^3$, and consider the convex polyhedra obtained by removing lattice points from $\Delta_{\mathbb{Z}}$. For example, removing the origin from the octant leaves a convex polyhedron, or removing the origin and a neighbor (shown in the figure below).



This is the same as removing stacks of boxes in Δ. We define the partition function

$$Z_\Delta(q) = \sum_{\text{stacks of boxes in}\Delta} q^{\#\text{ of boxes}}.$$

At $q \to 1$, the configuration that dominates the partition function is determined by a Riemann surface σ_D. How can we describe this limit shape? For a corner of a room, it looks like this:



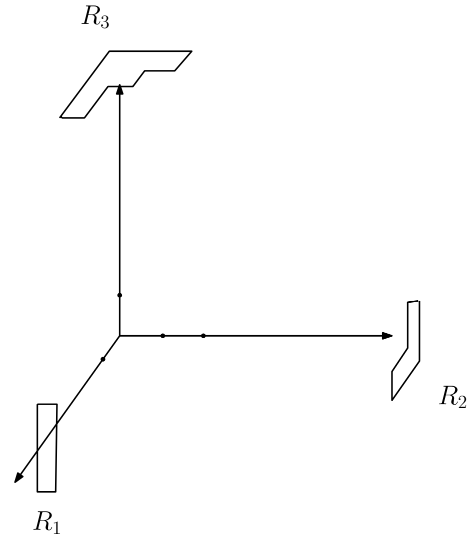
and is the solution of $P_\Delta = e^x + e^{-p} - 1 = 0$, where x and p are complex numbers. We would like to obtain such a function P_Δ for general Δ. We define

$$R(e^x, e^p) = \int d\theta\, d\phi |\log \mathbb{P}_\Delta(e^{x+i\theta}, e^{p+i\varphi})|.$$

More generally, we can define R_i to be two dimensional partitions. Then

$$Z_\Delta(q) \to Z_{R_1,\ldots,R_n,\Delta}(q),$$

where n is the number of infinite edges of Δ.



Claim:

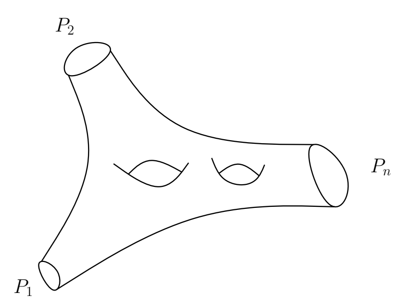
$$Z_{\Delta,R_1,\dots,R_n}(q)/Z_\Delta = \hat{Z}_{R_1,\dots,R_n,\Delta}(q).$$

The classical $W_{1+\alpha}$ algebra is $W_{n,m} = e^{nx}e^{mp}$, $\{p,x\} = 1$. Note that $\hat{Z}_{\Delta,R_1,\dots,R_n}$ has a $W_{1+\infty}$ symmetry that can be used to find it. Note that the set of two-dimensional partitions $\{R\}$ is equal as a Hilbert space to theory of a free fermion with $Nf = 0$.

Consider the quantum algebra acting on $\mathbb{Z}_\Delta$

$$\psi(x) = \sum_{n\in\mathbb{Z}+1/2} \psi_n e^{nx}(dx)^{1/2}\psi^*(x) = \sum \psi^*_n e^{nx}(dx)^{1/2},$$

where $\{\psi_n, \psi^*_m\} = \delta_{n+m,0}$. Recall our Riemann surface $P_\Delta(e^x, e^p) = 0$.

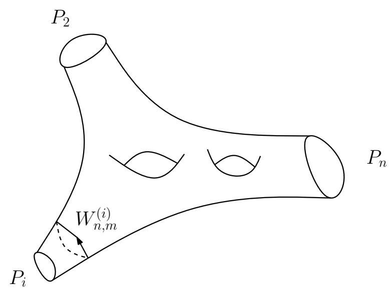


At each puncture P_i, choose a set of coordinates (x_i, p_i) for which $x_i \to \infty$ at P_i and $p_i = 0$ at P_i. Then H_i of P_i is a Hilbert space

$$\psi^{(i)}(x_i) = \sum_{n\in\mathbb{Z}+1/2} \psi^{(i)}_n e^{nx_i}, (\psi^*)^{(i)}(x_i) = \sum_{n\in\mathbb{Z}+1/2} (\psi^*)^{(i)}_n e^{nx_i}.$$

Consider an action of

$$W^{(i)}_{n,m} = \oint \psi^{*(i)}(x_i) e^{nx_i} e^{mp_i} \psi^{(i)}(x_i).$$

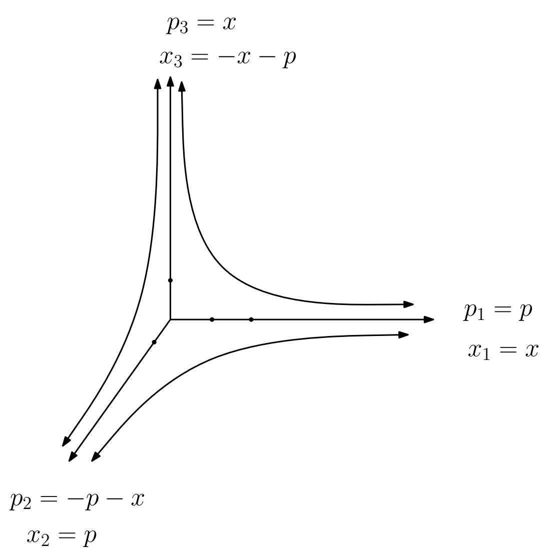


Then the symmetry of $|Z_\Delta\rangle$ is

$$W^{(i)}_{n,m} = -\sum_{j\neq i} W^{(j)}_{n_{i,j},m_{i,j}},$$

where $nx_i + mp_i = n_{ij}x_j + m_{ij}p_j + e_{ij}$. This explains how to translate the action of one $W_{1,+\infty}$ operator to the other punctures.

Example. Consider the octant again. Shown in the figure are the coordinates we associate with each puncture.



We get

$$\oint_{P_1} \psi_1(x_1)e^{nx_1}\psi_1(x_1) + \oint_{P_2} \psi_2(x_1)e^{nx_2-\hbar\partial_{x_2}}\psi_2(x_1) + \oint_{P_3} \psi_3(x_3)e^{n\hbar\partial_{x_3}}\psi_3(x_1) \tag{1}$$

Then $\mathbb{Z}_\Delta(q)$ satisfies (1) with $q = e^\hbar$. Locally, this theory is just the theory of free fermions.

Moreover, in the $\hbar \to 0$ limit, for $W_{1+\alpha}$ the Ward identities of α gives free boson CFT on Σ.

$$\oint \psi^* e^{nx} e^{mp}\psi \to \oint e^{nx}e^{mp}\partial\varphi,$$

as $\hbar \to 0$, where $p = p(x)$ is determined from $P_\Delta(x, p) = 0$ and $\psi(x) = e^{\varphi(x)/n}$, and $\varphi(x)$ is a free boson.

For $\hbar \neq 0$ the theory differs from free CFT in that under change of coordinates on Σ, $(x, p) \mapsto (x', p')$ gives

$$\psi(x) \to \int e^{S(x,\lambda')/\hbar} \psi(x) = \psi(\tilde{x}'),$$

with $dS = p\, dx - p'\, dx'$.

The structure is general:

(a) $c = 1$: $H(x, p) = xp = \mu = 0$, $[x, p] = \hbar$.

(b) (r, s) minimal models coupled to growth $H(x, p) = p^r + x^s$.