The nested loop approach to the *O(n)* model on random maps

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Notes taken by Samuel S Watson

Many models can be reformulated in terms of "loop gases." For example, $O(n)$ loop models where n plays the role of a loop fugacity.

This model is naturally defined on random maps (a.k.a. dynamical random lattices). The partition function is

$$Z = \sum_{\text{graphs}} Z_{O(n)}(\mathcal{G}).$$

This solution consists in the computation of the partition function and other "global" quantities, but little is known on the "local" geometry.

For many map ensembles (random planar graphs with a plane embedding, modulo continuous deformation), the typical graph distance between vertices scales at $m^{1/4}$, where m is the map size. The scaling limit is the Brownian map (Le Gall, Miermont).

One shortcoming is the difficulty of extending this construction to physical models incorporating matter. We will show that certain maps with "large faces" (studied by Le Gall and Miermont) are related to $O(n)$ loop models.

A rooted planar map is a graph embedded in the plane, considered up to continuous deformation, with a distinguished root edge incident to the outer face.

Natural probability measures over maps: uniform distribution over maps with m edges, over triangulations with m triangles, over quadrangulations with m squares, etc. In fact, there is a bijection between the first and third of these sets, though they are not the same sets if one has in mind an application for which the graph distance is important.

More generally, we can control the degrees of the faces by defining

$$\mathcal{F}_p(g_1, g_2, \ldots) = \sum_{\text{maps}} \prod_{k \geq 1} g_k^{\#\text{ faces with degree } k}$$

We define such a sequence $(g_1, \ldots)$ to be admissible if $\mathcal{F}_p(g_1, g_2, \ldots) < \infty$ for all p. If $\mathbb{P}(\#\text{ vertices} > m)$ decays exponentially, we call the sequence subcritical; if polynomial we call the sequence critical. [generic and non-generic]

Pick a sequence $(g_1^\circ, \ldots)$ such that $g_k^\circ \sim k^{-a}$, for $a \in (3/2, 5/2)$. There exist unique constants A, B such that the weight sequence $g_k := AB^k g_k^\circ$ is non-generic critical and then

$$\mathbb{P}(\text{ degree of a typical face } > k) \sim (\text{const}) k^{-a+1/2}.$$

We consider loops on the dual map by visiting all the faces of the map and defining loops to be the traversed edges. We don't visit the outer face. Each configuration receives a weight $n^{\#\{\text{loops}\}} \times$ local weights.

We cut along the outer and inner contours of a given loop (these sets consist of the vertices incident to some edge intersecting the loop. We call this a *necklace* (i.e., a sequence of polygons glued side-by-side).

The outer component is called the *gasket*. It is a map without loops, with the same outer degree as the original map. What we obtain is a bijection between configurations and triples of the form (gasket, necklaces, internal configurations). For this to work, we need some natural compatibility conditions on these triples. Each gasket is a map whose faces are either regular faces or holes. Each hole of degree $k \geq 1$ is associated with a necklace of outer length k. Each necklace of inner length $k' \geq 0$ is associated with an internal configuration of outer degree k'.

Proposition 1. The partition function of our $O(n)$ loop model is obtained from the generating function for maps with controlled face degrees via $F_p = \mathcal{F}_p(g_1, \ldots)$, where the g_k's satisfy the fixed point condition

$$g_k = g_k^{(0)} + n \sum_{k' \geq 0} A_{k,k'} \mathcal{F}_{k'}(g_1, g_2, \ldots).$$

If we consider a critical loop model, then we get a corresponding non-generic weight sequence.

Suppose that we want to count the number of necklaces on map. If the map is a triangulation, we get a nice rational function for $A_{k,k'}$. On quadrangulations the picture is not quite as nice:

$$A_{k,k'} = \sum_{j \equiv k \bmod 2} \frac{2k}{k+k'} \binom{(k+k')/2}{j, (k-j)/2, (k'-j)/2} h_1^j h_2^{(k+k')/2-j}.$$

More generally, we may consider arbitrary face weights, with somewhat less concrete formulas for A.

We define the resolvents

$$\mathcal{W}(x) = \sum_{p \geq 0} \frac{\mathcal{F}_p(g_1, g_2, \ldots)}{x^{p+1}}$$

and

$$W(x) = \sum_{p \geq 0} \frac{F_p}{x^{p+1}},$$

where script letters are used for maps with controlled face degrees, and plain letters are used for the $O(n)$ loop model. The one-cut lemma tells us that for any admissible sequence $(g_1, \ldots)$, $\mathcal{W}$ defines an analytic function on $\mathbb{C} \setminus [\gamma_-, \gamma_+]$. The spectral density is

$$\rho(x) = \frac{\mathcal{W}(x + i0) - \mathcal{W}(x - i0)}{x}.$$

In fact, the resolvent is determined by $\mathcal{W}(x+i0) + \mathcal{W}(x-i0) = x - \sum_{k \geq 1} g_k x^{k-1}$, and the condition $\mathcal{W}(x) \sim 1/x$ as $x \to \infty$.

Recall that for the $O(n)$ loop model on triangulations, $A(x,y) = hx/(1-h(x+y))$, and on quadrangulations

$$A(x,y) = \frac{h_1xy + 2h_2x^2}{1 - h_1xy - h_2(x^2+y^2)}.$$

Suppose that $A(x,y)$ is rational with a single pole in y at $y = s(x)$ (as in the triangular and rigid quadrangular cases). In/out symmetry implies that s is a homographic involution.

$$s(x) = \frac{\alpha - \beta x}{\beta - \delta x}.$$

This situation is generically realized in a model with loop bending energy. We introduce a conformal mapping to the torus. The functional equation satisfied by W becomes

$$\omega(\nu + iT') + \omega(\nu - iT') = n\omega(\nu),$$

with ω odd and $2T'$ periodic. This can be done explicitly in terms of elliptic functions.

Going back to the x plane, this implies that W has a dominant singularity of the form $W(x) \propto (x - \gamma_+)^{1\mp b}$, as $x \to \gamma_+^+$. Hence

$$\mathbb{P}(\text{degree of a typical gasket face} > k) = (\mathsf{const})k^{-a+1/2}.$$

In summary, we have shown that the gasket of a critical $O(n)$ loop model has anon-generic critical Boltzmann map distribution. The corresponding stable map has Hausdorff dimension $3 \pm \frac{2}{\pi}\arccos(n/2)$, for $n \in (0,2)$.

Question: Correlations of operators in this model?

Answer: No idea.

Question: Can you say something in the case of FK random cluster models?

Answer: Yes, the work applies to both dense and dilute phases (corresponding to CLE_κ with $\kappa > 4$ and $\kappa \leq 4$, respectively).

Question: Which powers of m are obtained in these models?

Answer: Exponents between 1/4 and 1/2.