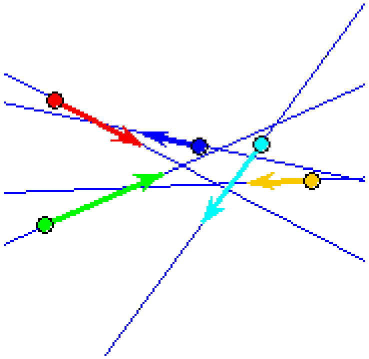


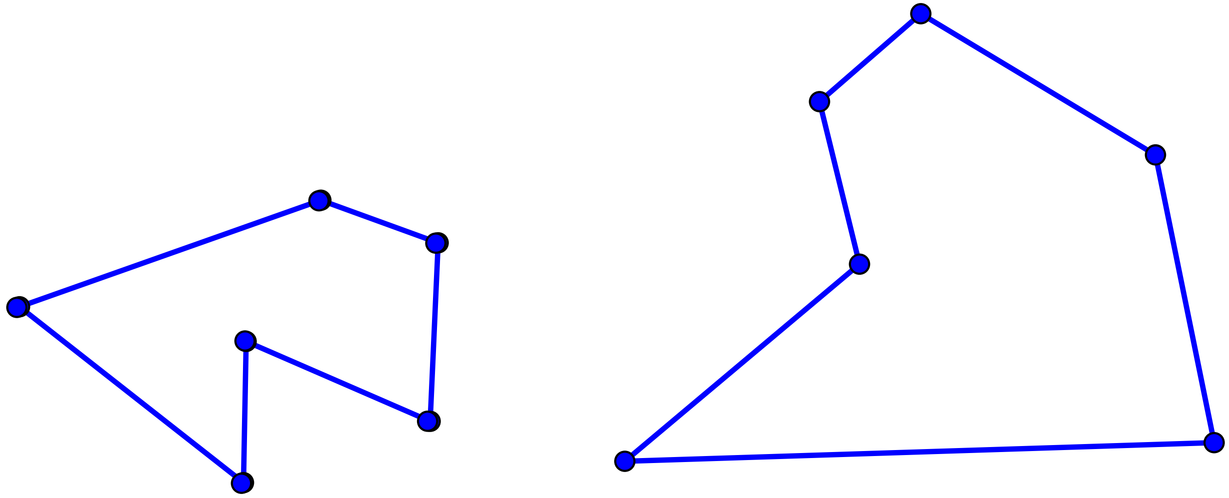
Points in Motion

Ileana Streinu

Smith College

Motivation 1:

Morphing two polygonal shapes



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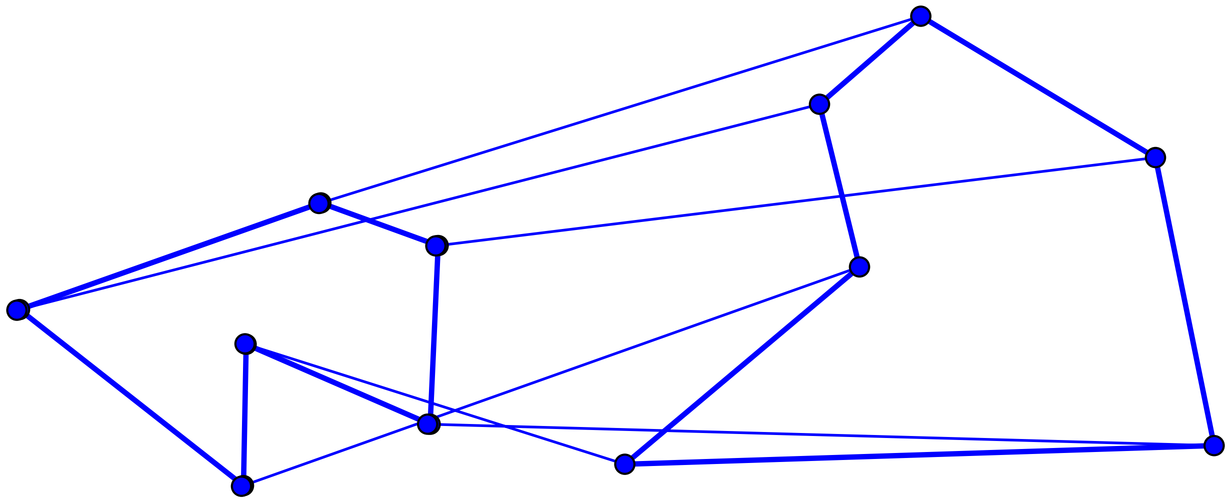

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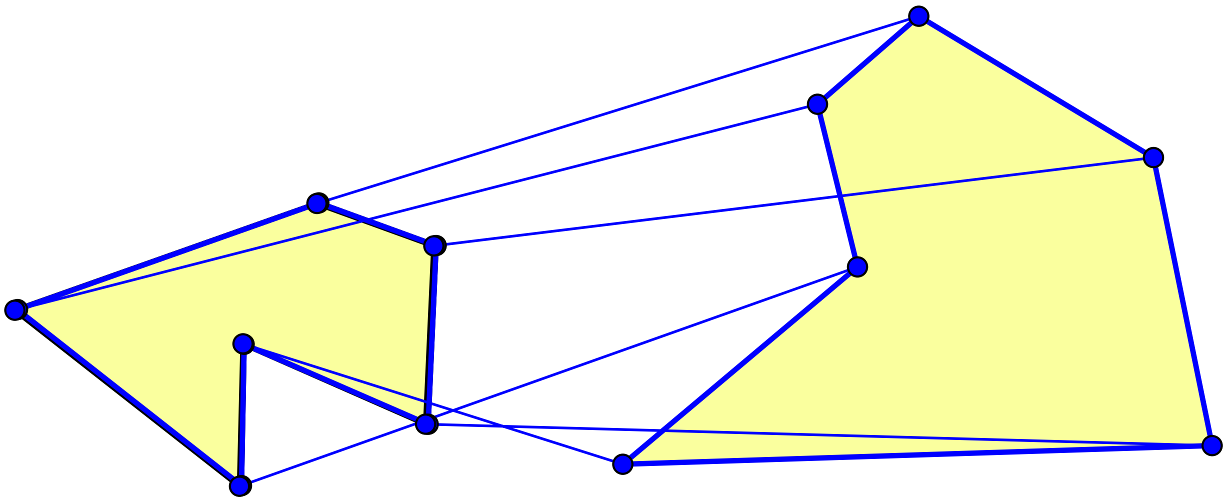

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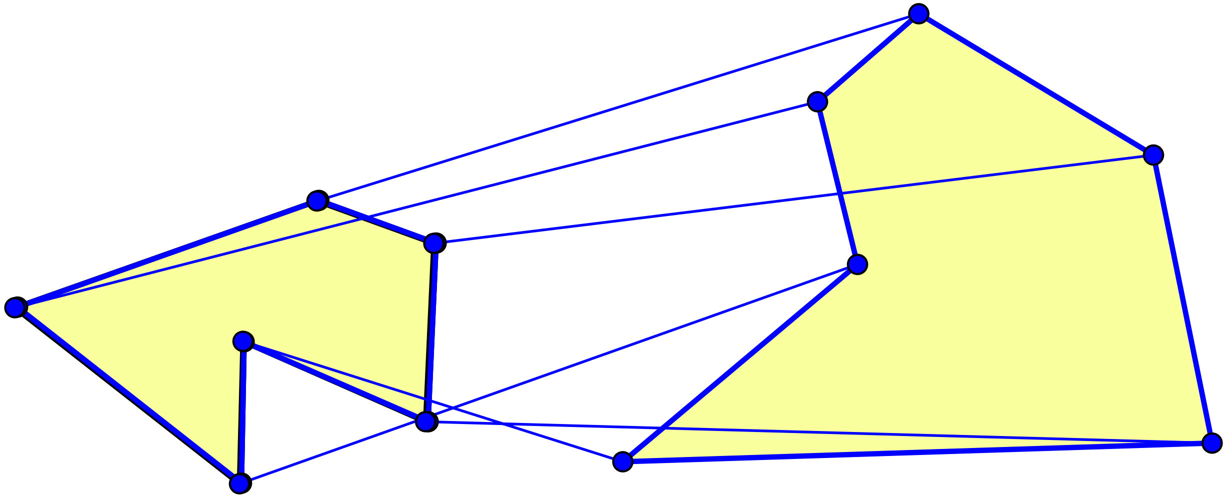

Points move with constant velocities

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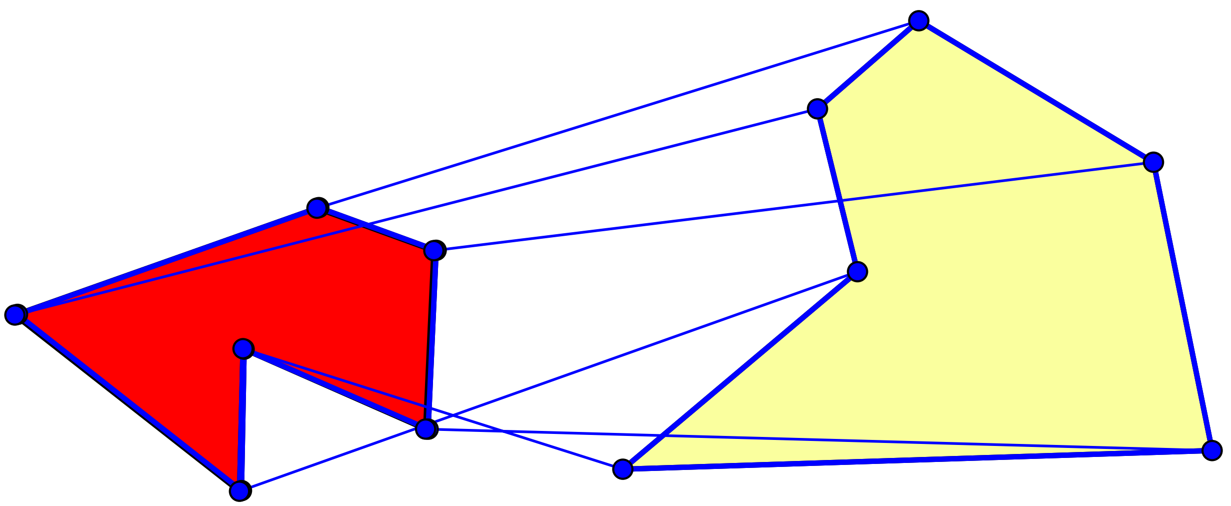

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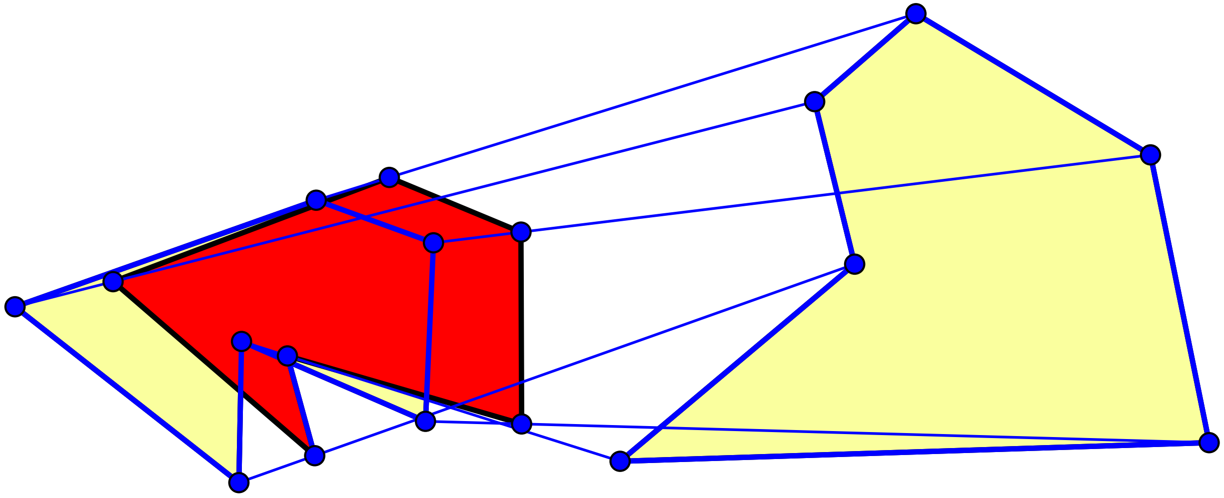

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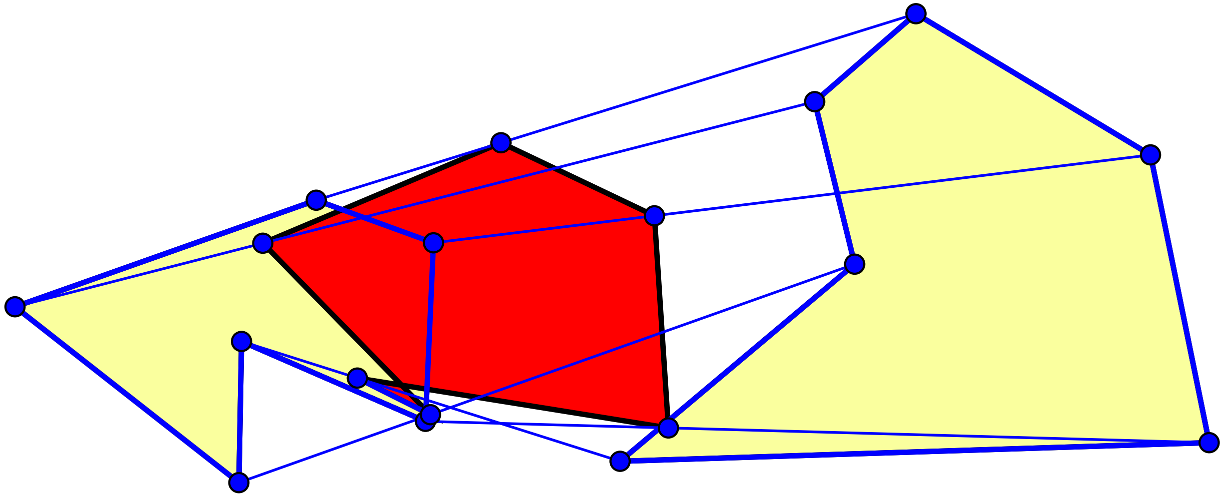

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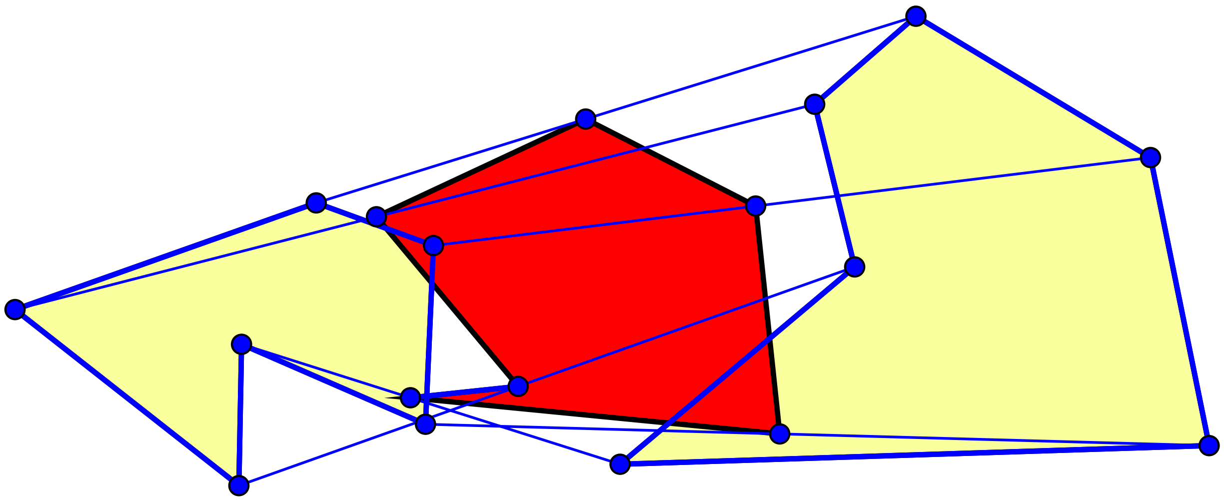

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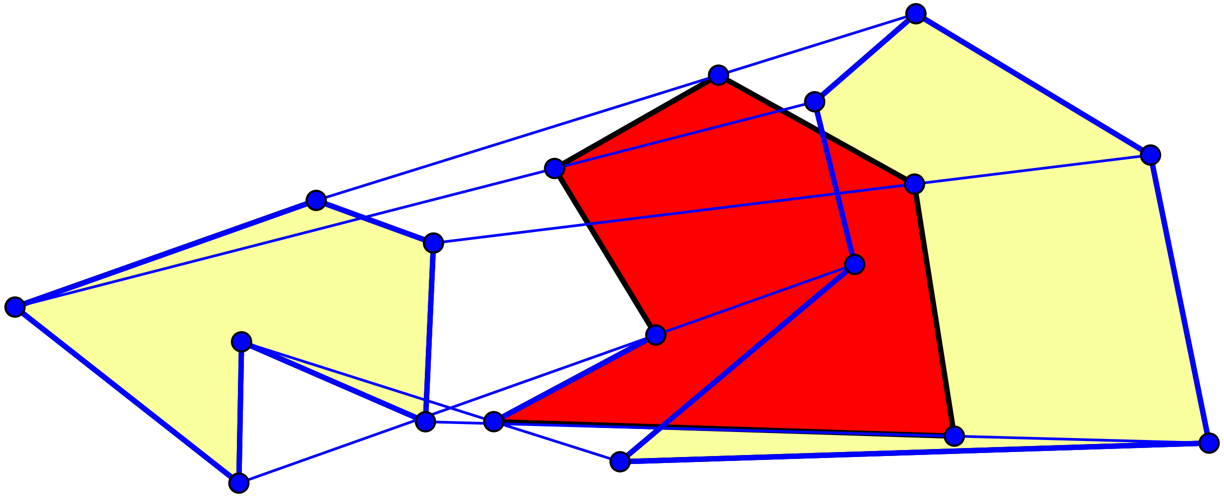

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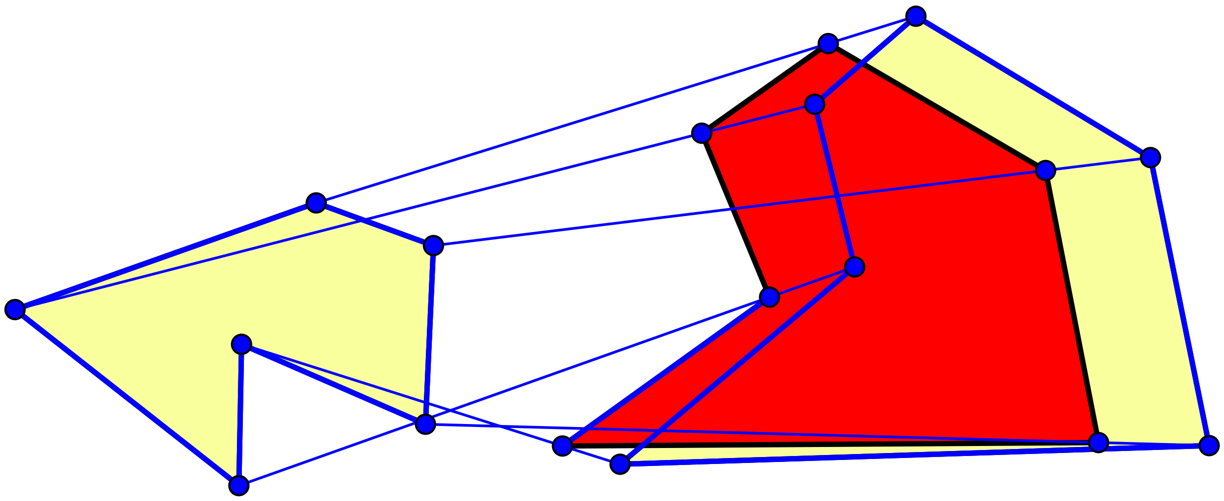

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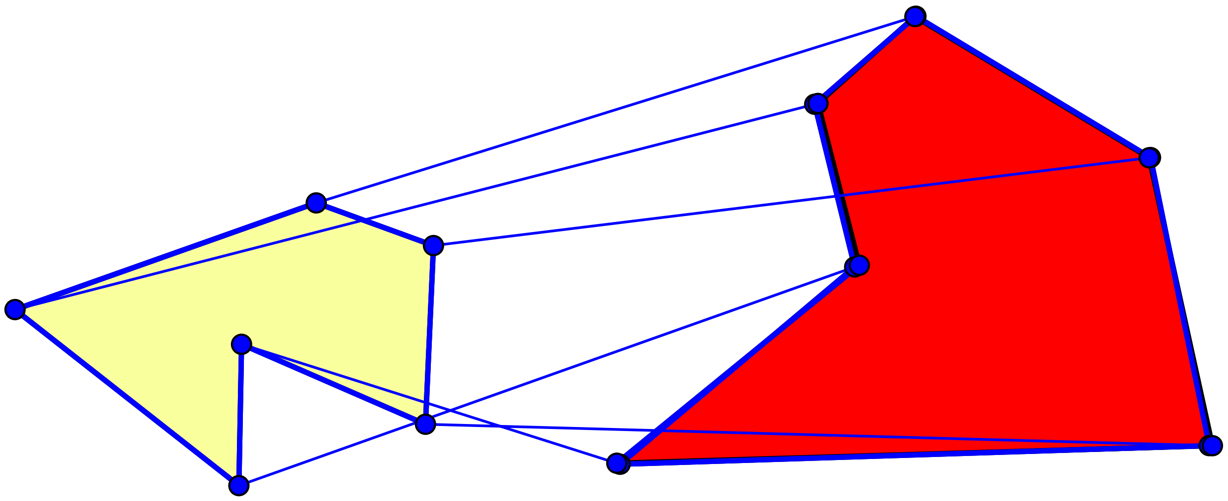

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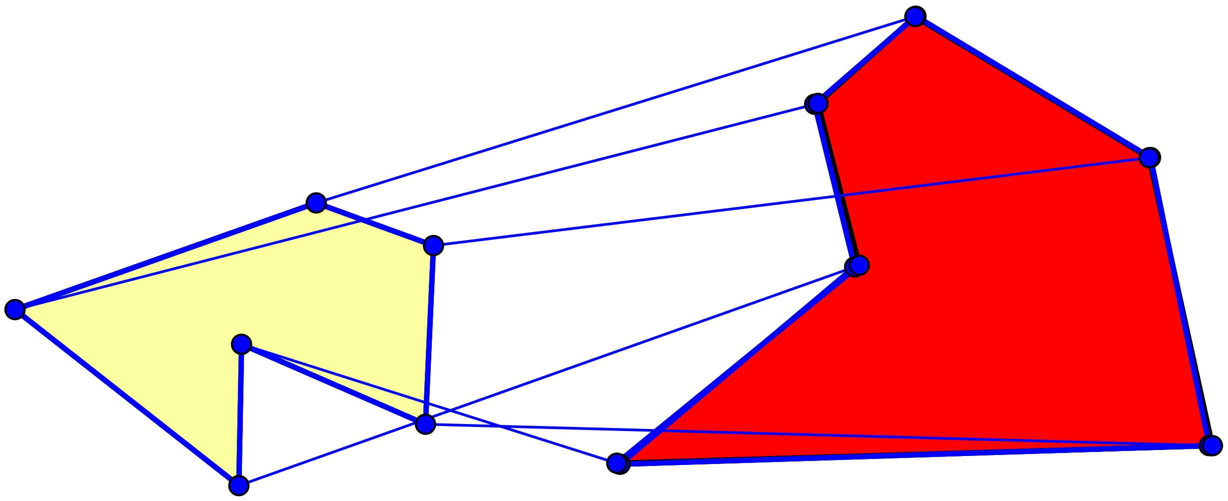
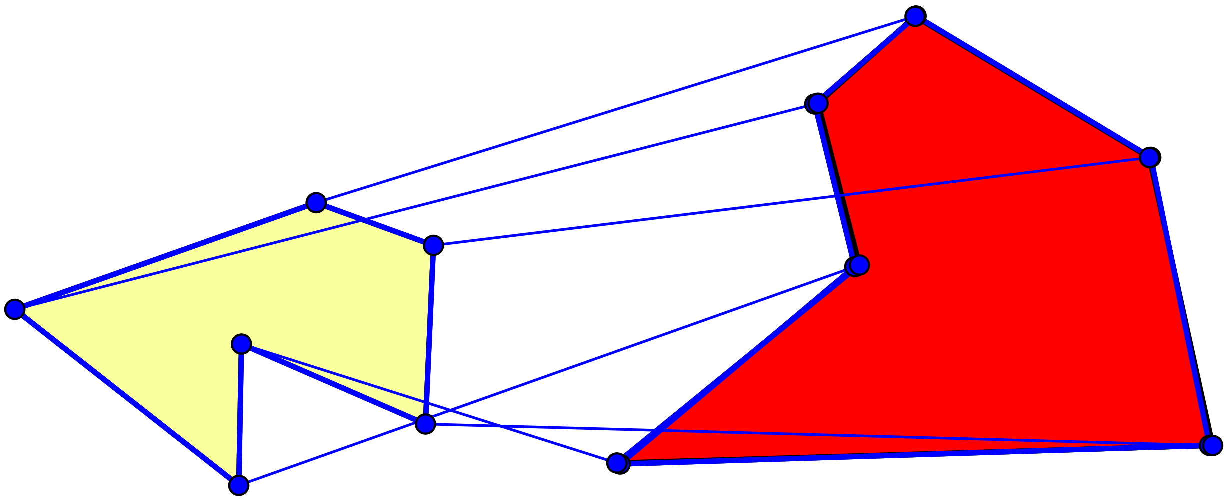


Question: when can we GUARANTEE simplicity?

Motivation 1:

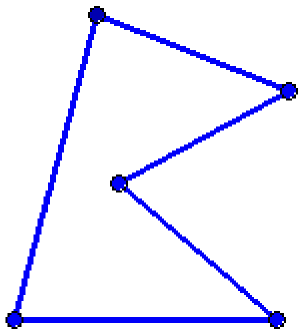
Morphing two polygonal shapes



Question: when can we GUARANTEE simplicity, for any duration of the constant-velocity motion?

Motivation 2:

Morphing simple polygons with parallel edges (Guibas, Hershberger, Suri'00)



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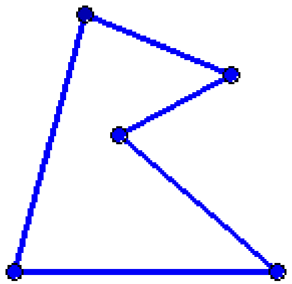

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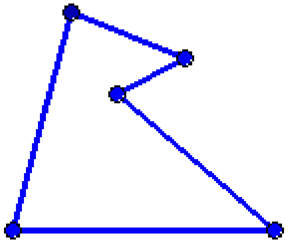

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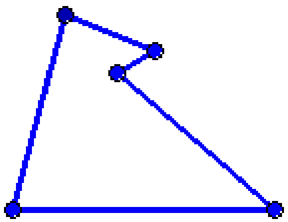

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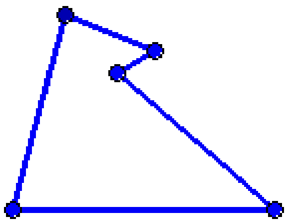

The motion is divided into pieces which are constant velocity, guaranteed to be collision free, and maintain the edge directions.

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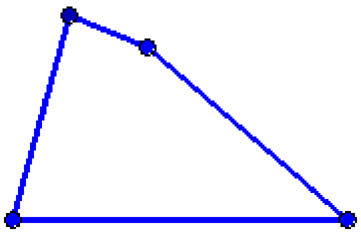
Morphing simple polygons with parallel edges (Guibas, Hershberger, Suri'00)



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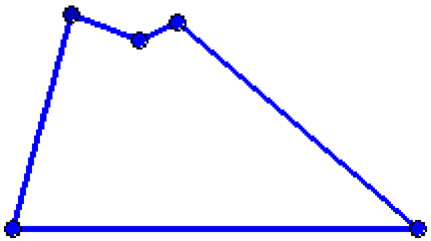

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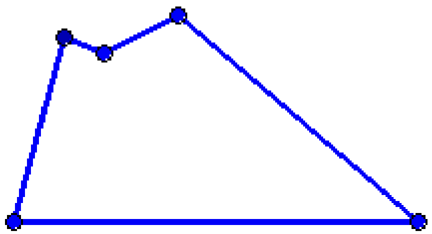

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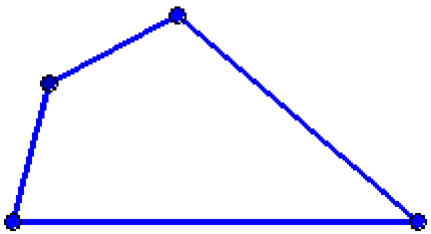

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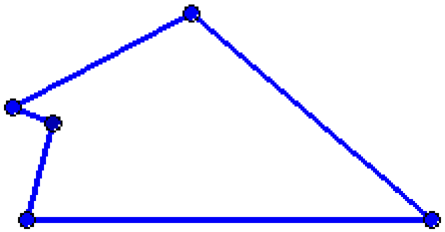

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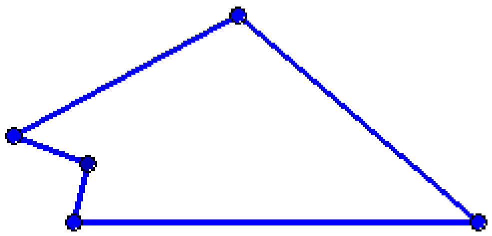

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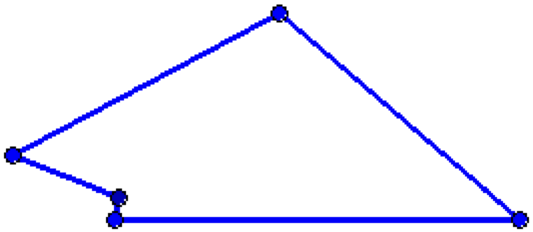
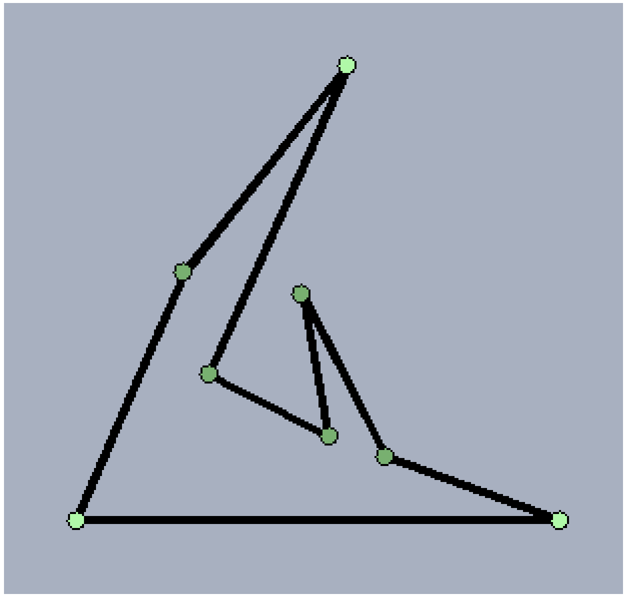


The motion is divided into pieces which are constant velocity, guaranteed to be collision free, and maintain the edge directions.

When can the whole morph be done with just **one** such motion?

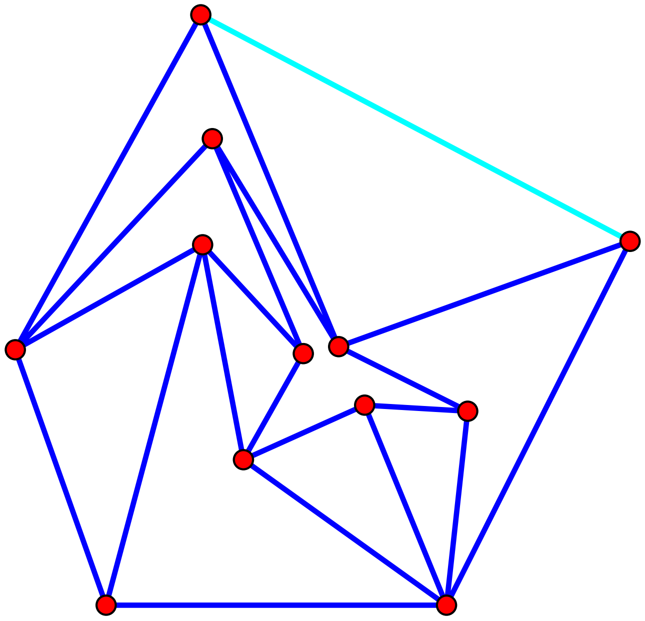
Example:

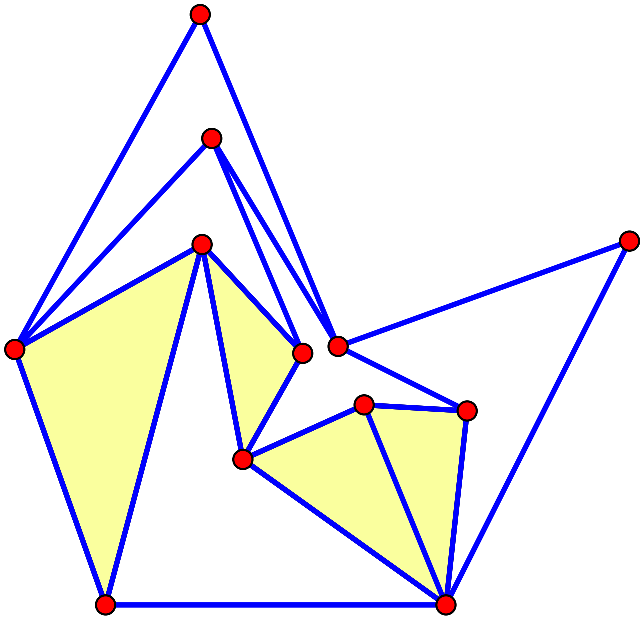


When can the whole morph be done with just one such motion?

My real motivation ☺

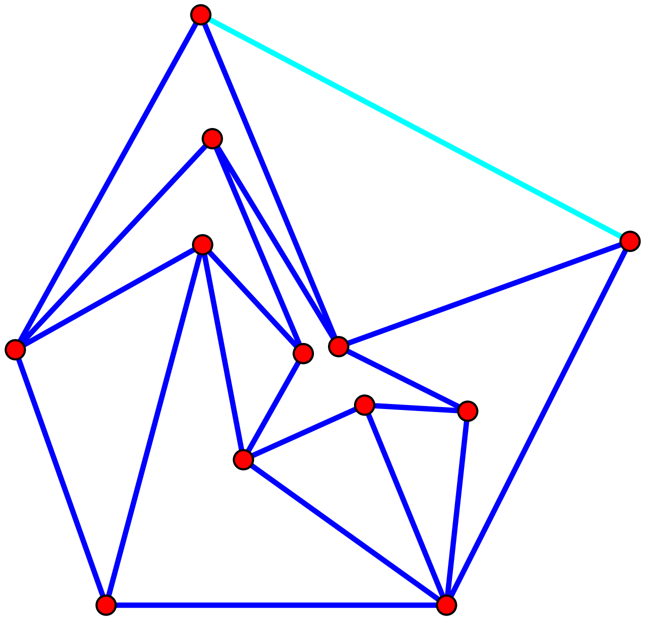
Pointed pseudo-triangulation mechanisms (S'00)

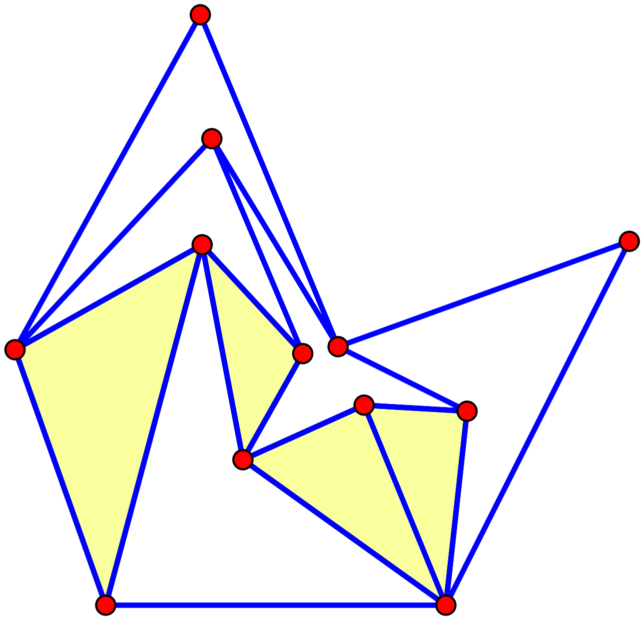




My real motivation ☺

Pointed pseudo-triangulation mechanisms (S'00)

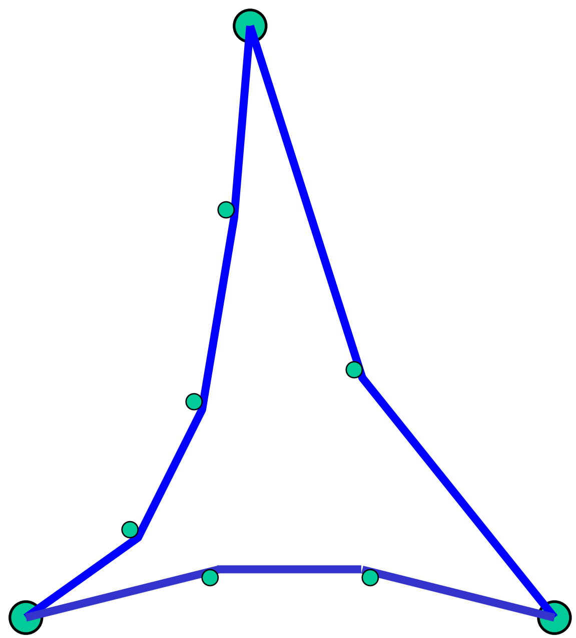




Pointed Pseudo-Triangulations: Definitions

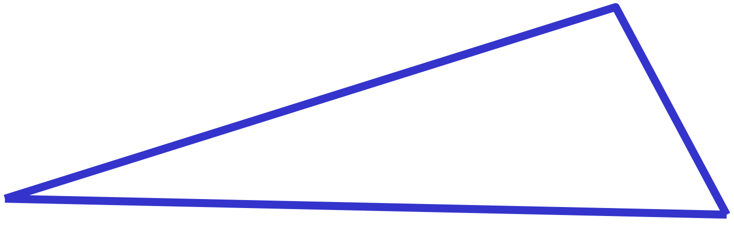
- Pseudo-Triangle
- Pointed Set of Edges
- PseudoTriangulation
- Pointed Pseudo-Triangulation

Pseudo Triangle



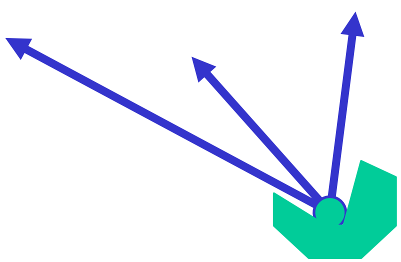
A **simple polygon** which has exactly **three inner convex vertices.**

In particular, a triangle is a pseudo-triangle.

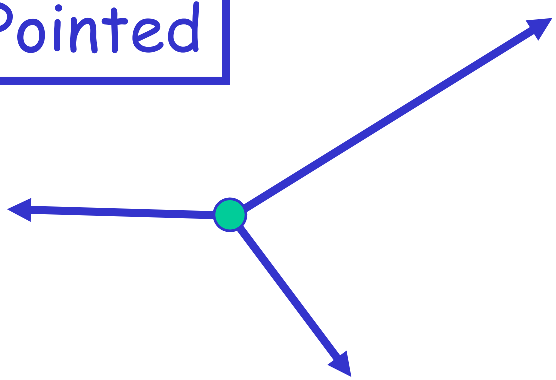


Pointed Planar Set of Vectors

Pointed

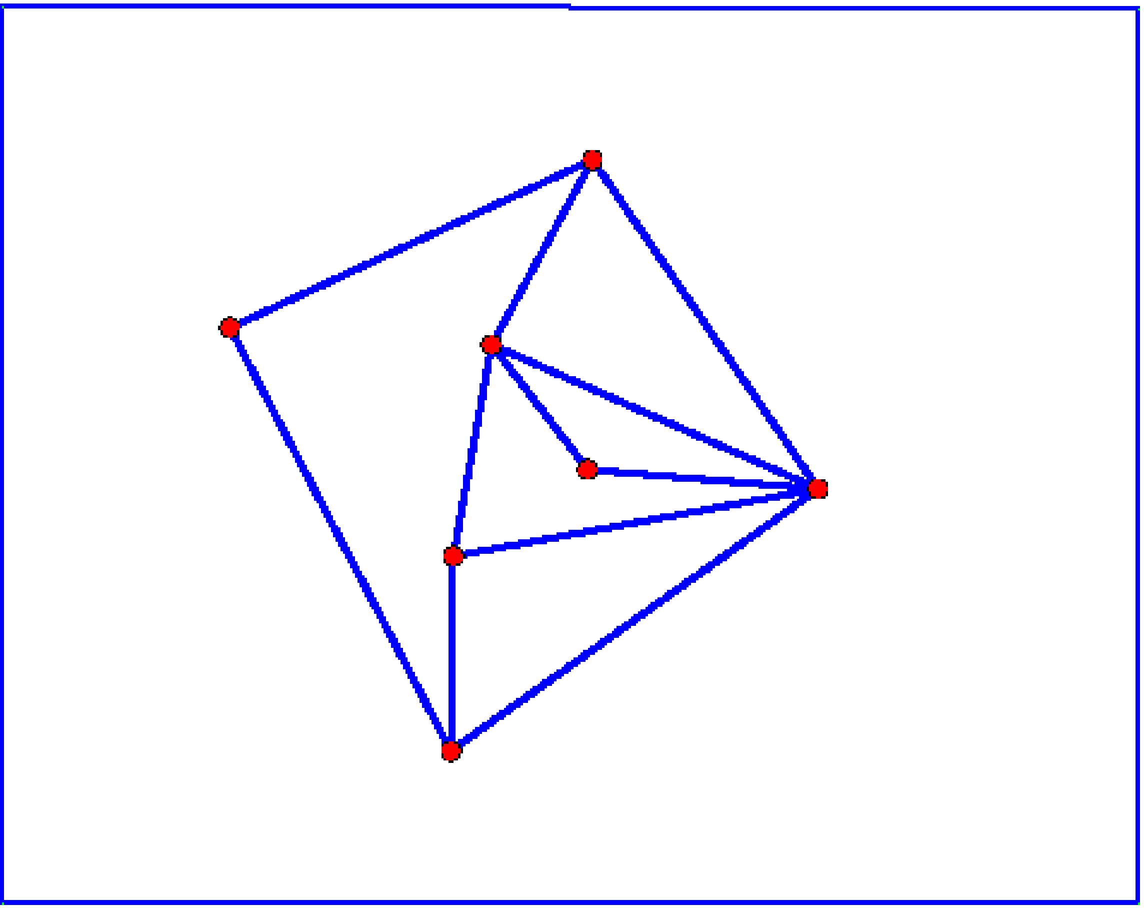


Not Pointed



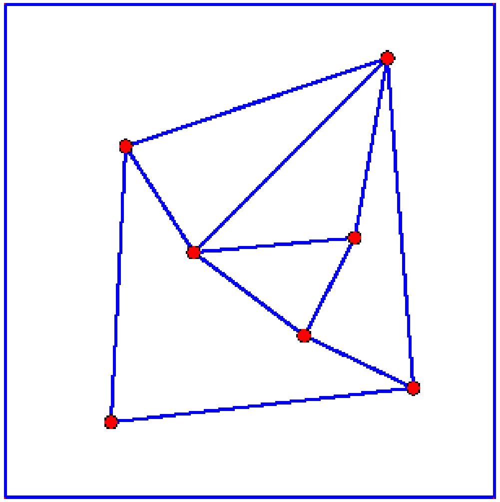
Two circularly adjacent vectors span a **reflex angle**

Pointed Pseudo Triangulation of a Planar Set of Points [S'00]

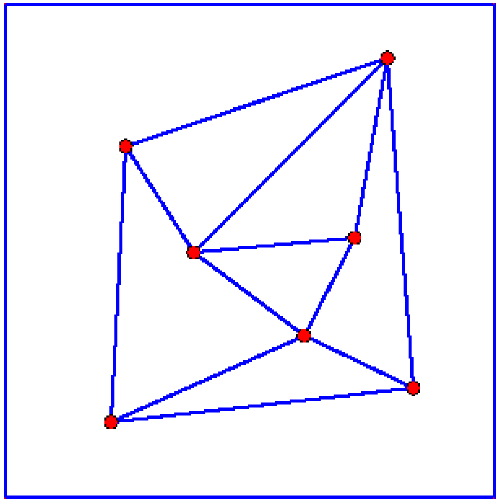


- Partitioning of the convex hull with a maximal set of non-crossing and pointed interior edges.
- The resulting faces are pseudo-triangles.

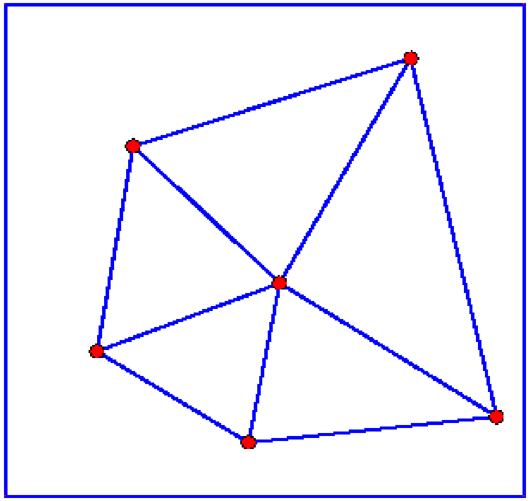
Other Pseudo Triangulations



Pointed



Not pointed



Not pointed

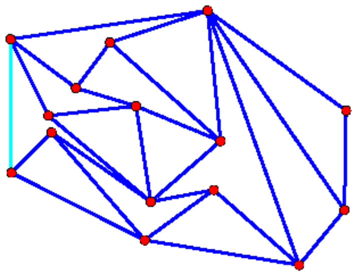
Pointed Pseudo Triangulations

Summary of main properties (S'00)

- Have exactly 2n-3 edges, n-2 faces
- Are pointed, and maximal with the property of being both planar and pointed.
- Have the hereditary Laman property: any subset of k vertices is planar, pointed and has <= 2k-3 edges
- Admit an inductive (Henneberg) construction.

Are minimally rigid graphs

(with a special embedding)

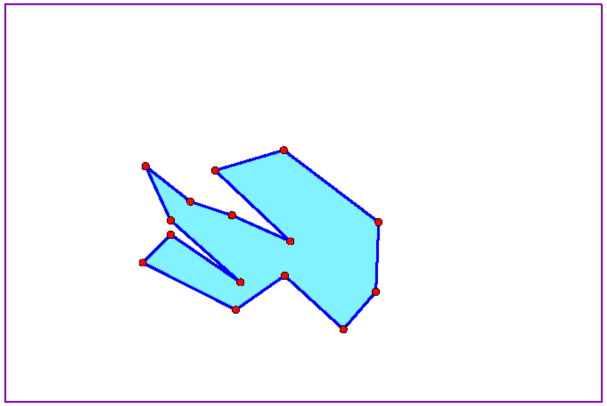


Main application of Pointed Pseudo-Triangulations

A solution to the

Carpenter's Rule Problem

The Carpenter's Rule Problem

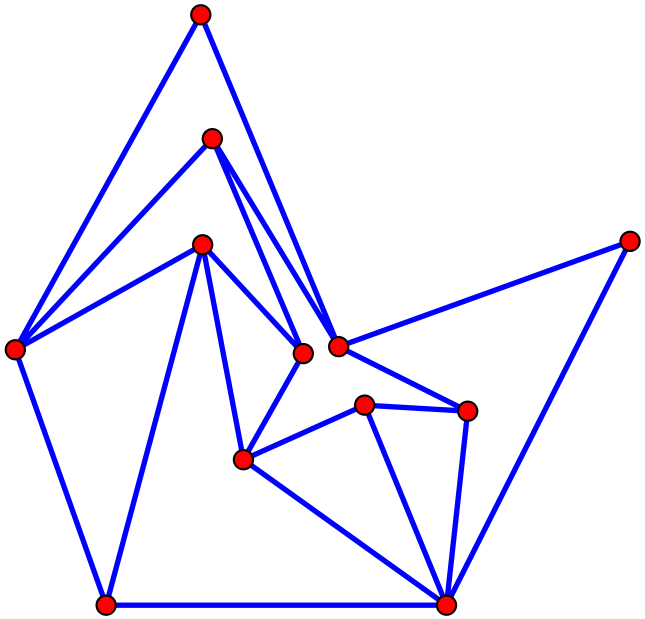


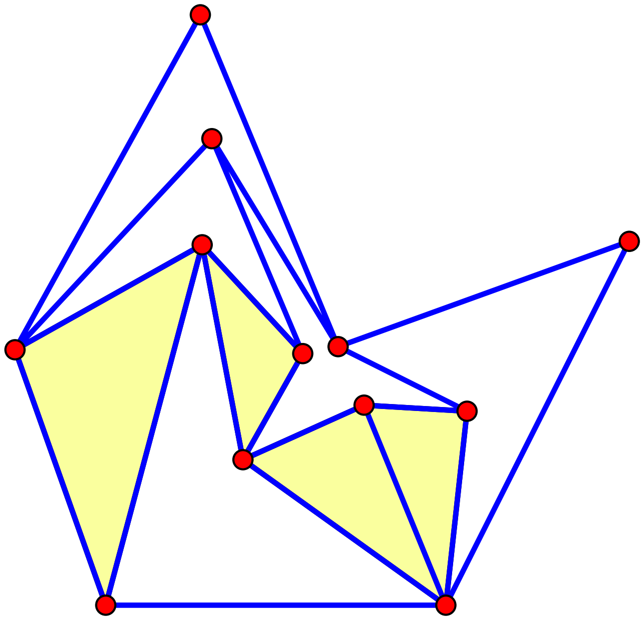
ODE-based:
Connelly, Demaine and Rote'00
See Erik Demaine's web page
http://theory.lcs.mit.edu/~edemaine

Pseudo-triangulation based:
(S'00)
From my web page
http://cs.smith.edu/~streinu

My real motivation ☺

Pointed pseudo-triangulation mechanisms (S'00)





As fixed-edge length mechanisms: expansive

As parallel redrawing mechanisms: WHAT?

Why study them?

- Observed all ppt-mechanisms are planar (non-crossing) →
- Observed points move with constant velocities ←
- Polygons on top of ppt mechanisms morph without proper crossings

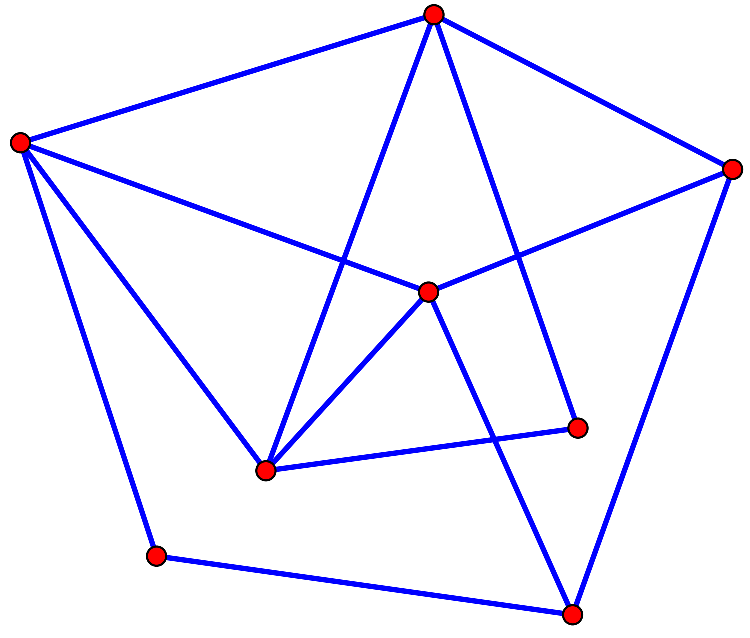
- Wanted to prove this ↓
- Ppt-mechanisms special case of 1dof Laman mechanism graphs: what about them? ↓
- Is kinetic planarity characterized by pointed pseudo triangulation mechanisms? ←
- → Apply it to other problems (morphing)

Overview:

- The Parallel Redrawing model of rigidity:
  - fixed edge-direction, rather than
  - fixed edge-length
- Objects of study:
  - 1dof Laman mechanisms
  - Pointed pseudo-triangulation mechanisms
- Restricted to **GENERIC** situations
- Kinetic objects
  - Points
  - Embedded graphs
  - Polygons
- Focusing on:
  - Collisions
  - Edge crossings
  - Combinatorial invariants:
    - Rigid components
    - Oriented matroidal invariants:
      - Partial hyperline sequences
      - Combinatorial Pseudo-triangulations
- Algorithms:
  - Parallel redrawing sweep
- Further directions:
  - Kinetic point sets
  - Kinetic graphs
  - Combinatorial parallel redrawing sweep for combinatorial pseudo-triangulation mechanisms

Background:

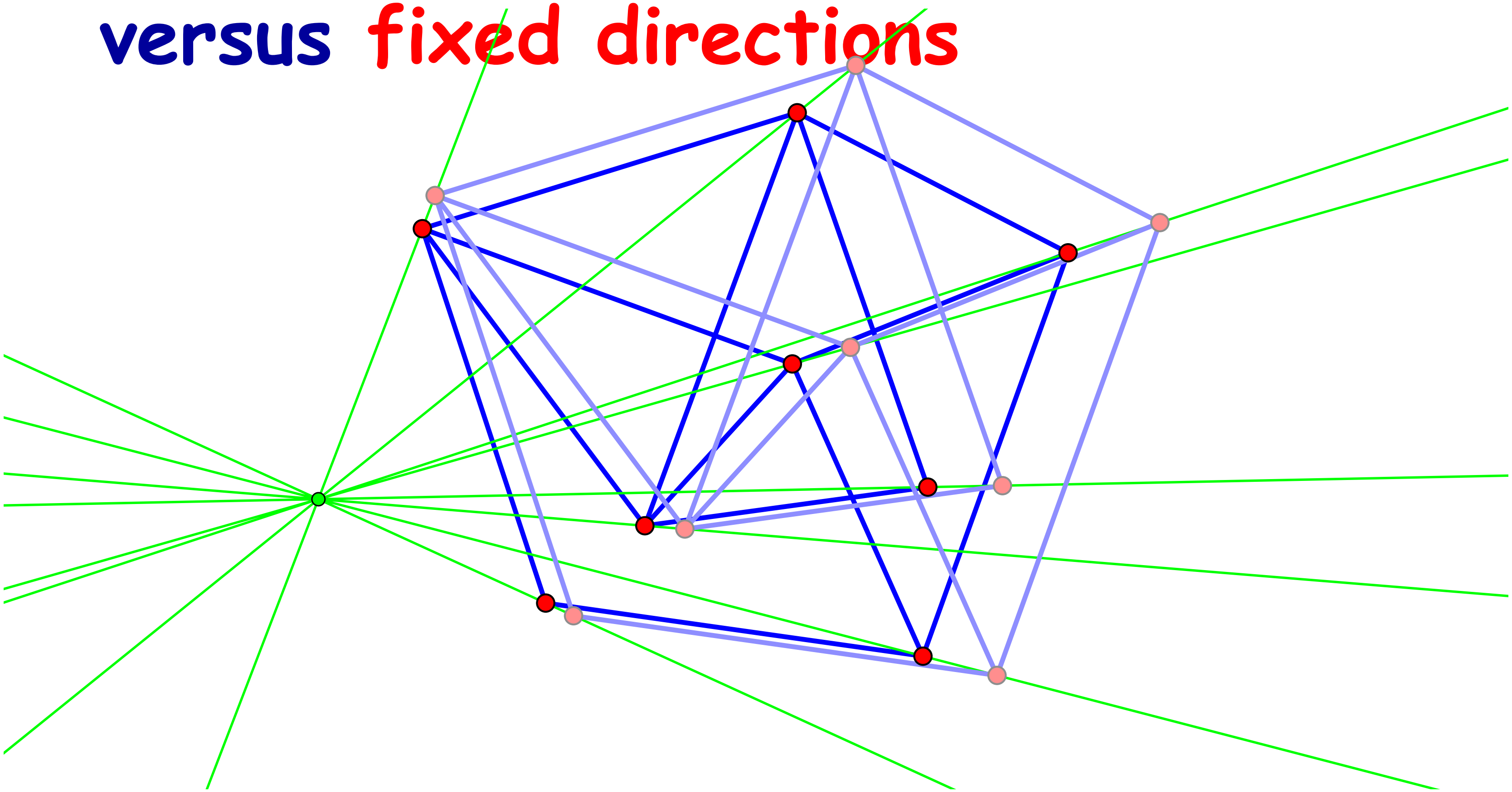
Rigidity with fixed edge-lengths versus fixed directions



Minimally Rigid (Laman) graph: 2n-3 edges,
every k-subset spans <=2k-3 edges

Background:

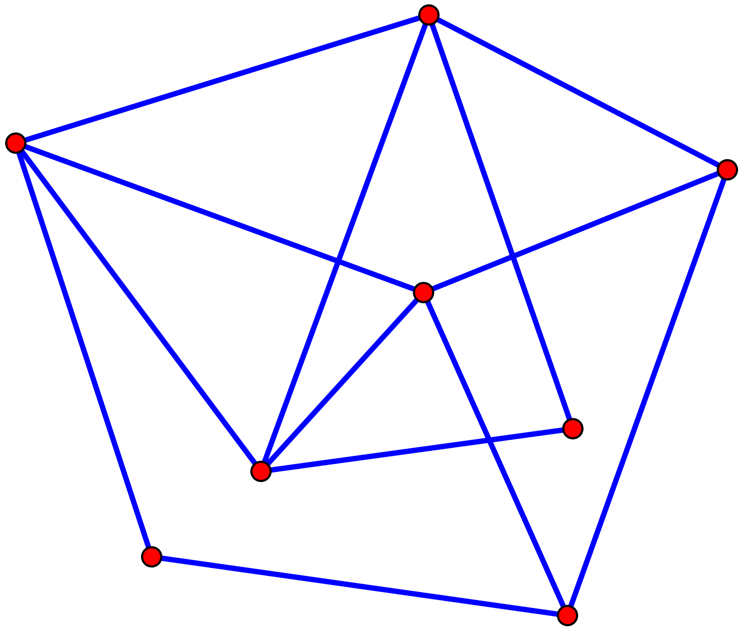
Rigidity with fixed edge-lengths versus fixed directions



Laman graphs have only trivial parallel redrawings

Background:

Rigidity with fixed edge-lengths versus fixed directions



Minimally Rigid

Background:

Rigidity with fixed edge-lengths
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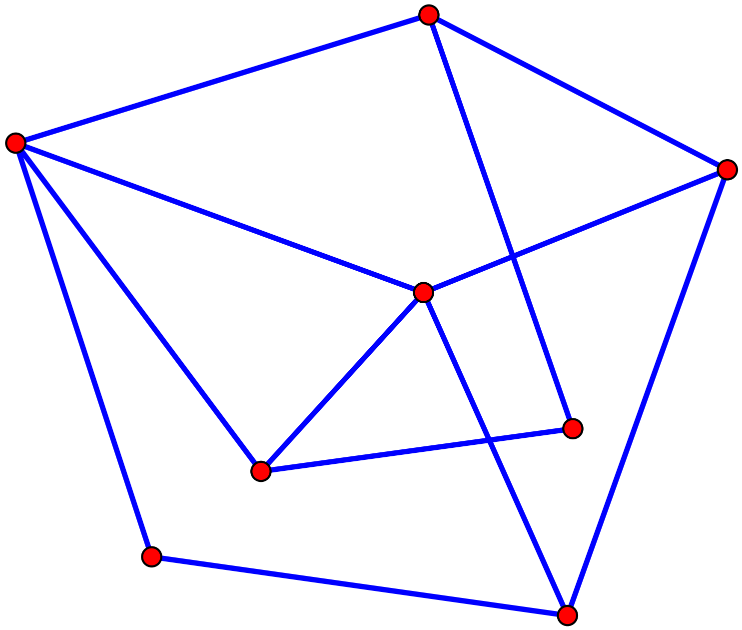
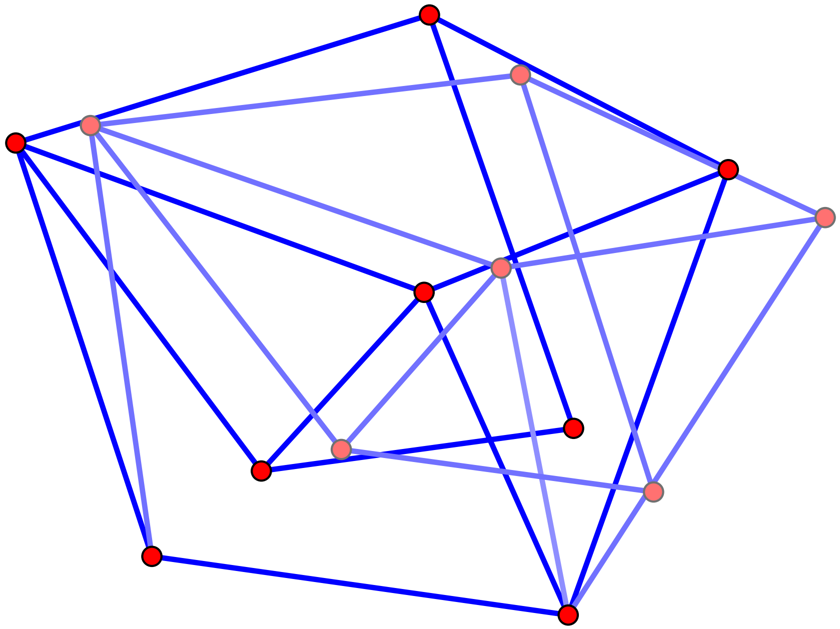


Laman mechanism

1dof flexible

Background:

Rigidity with fixed edge-lengths versus fixed directions



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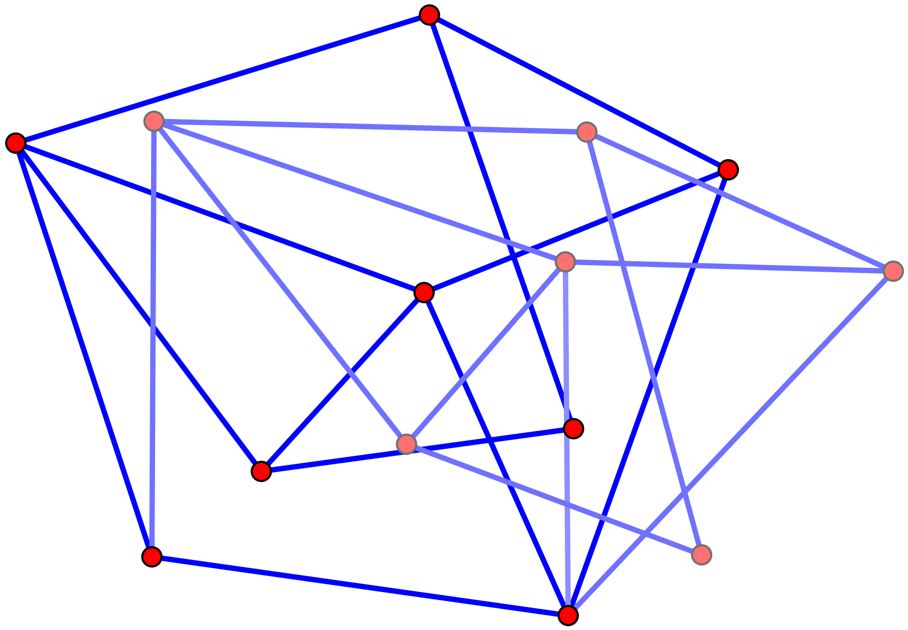

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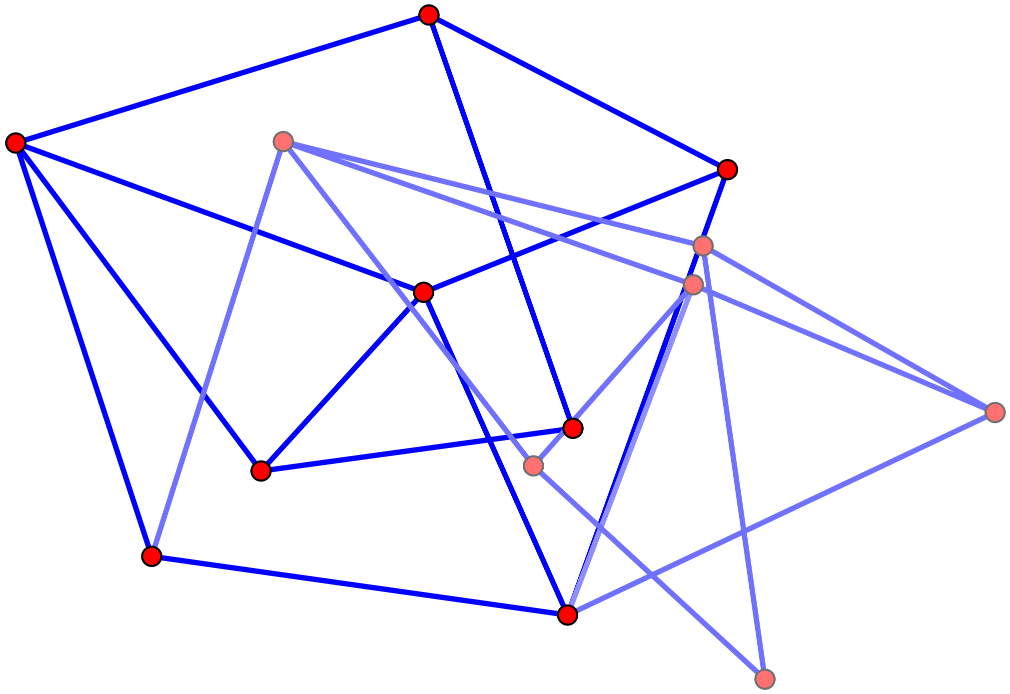

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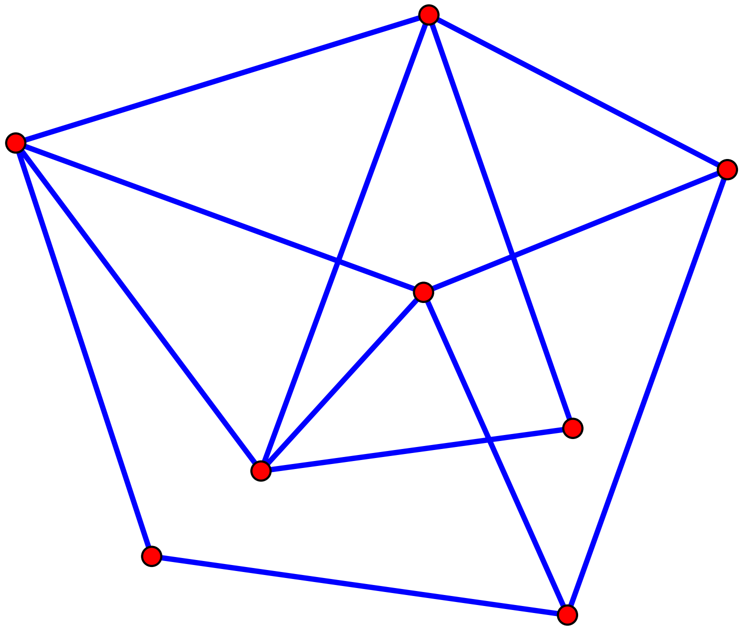
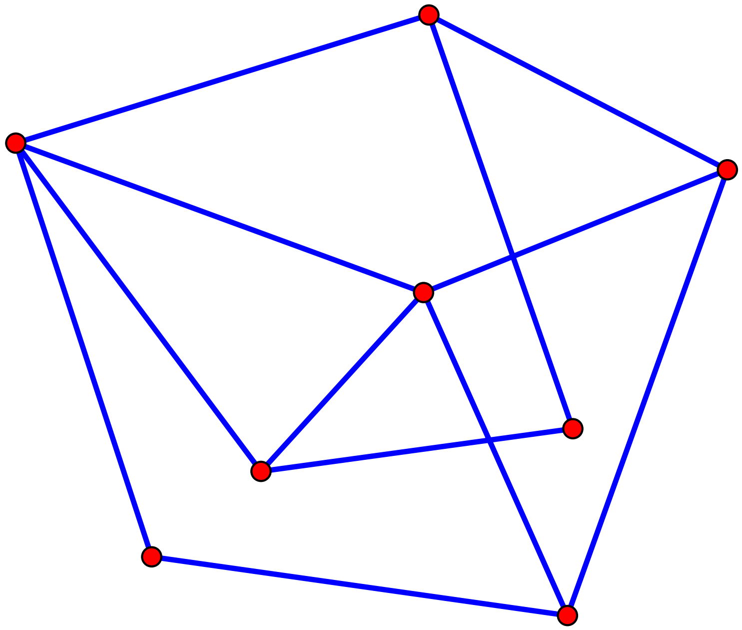


Laman graph:

only trivial parallel redrawings

Background:

Rigidity with fixed edge-lengths
versus fixed directions

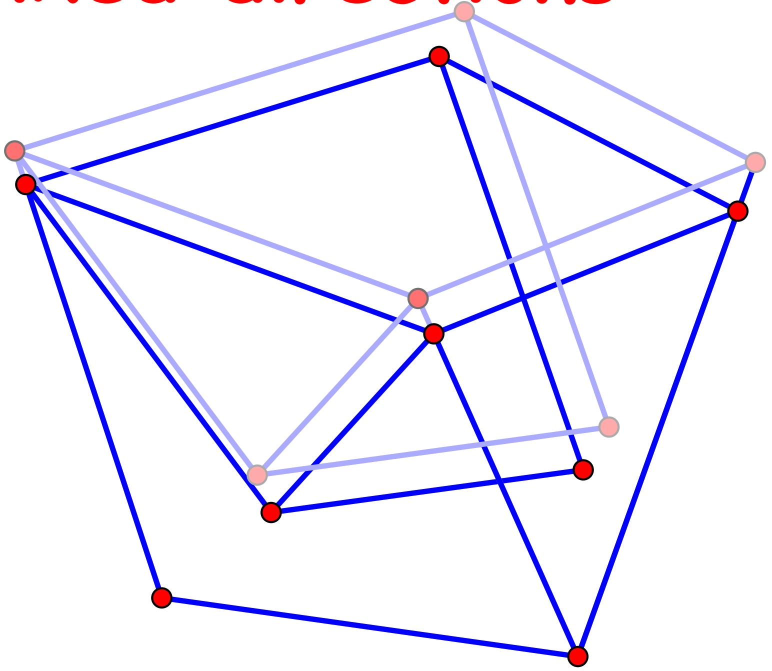


Laman mechanism (1dof flexible)

Non-trivial parallel redrawing

Background:

Rigidity with fixed edge-lengths versus fixed directions



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Non-trivial parallel redrawing

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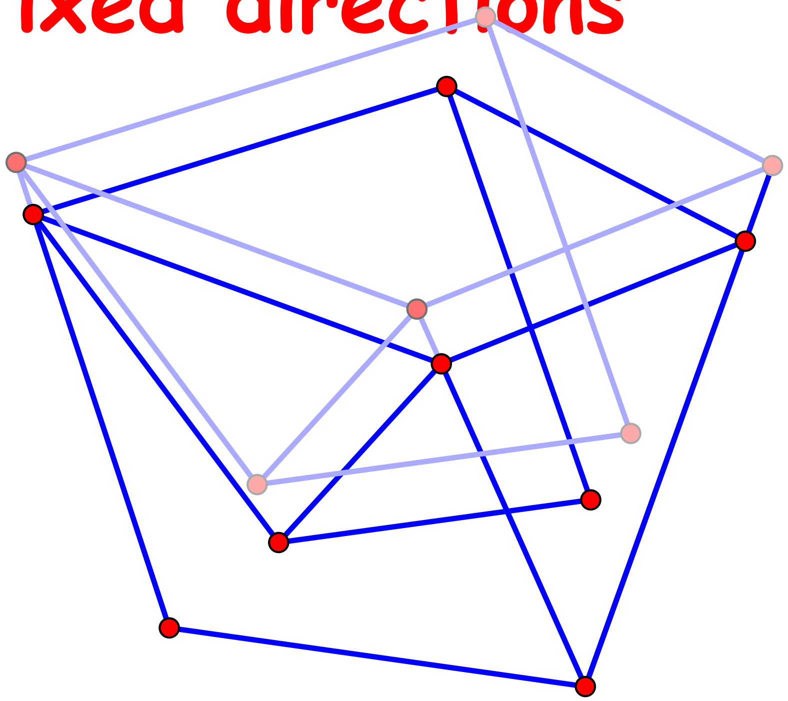

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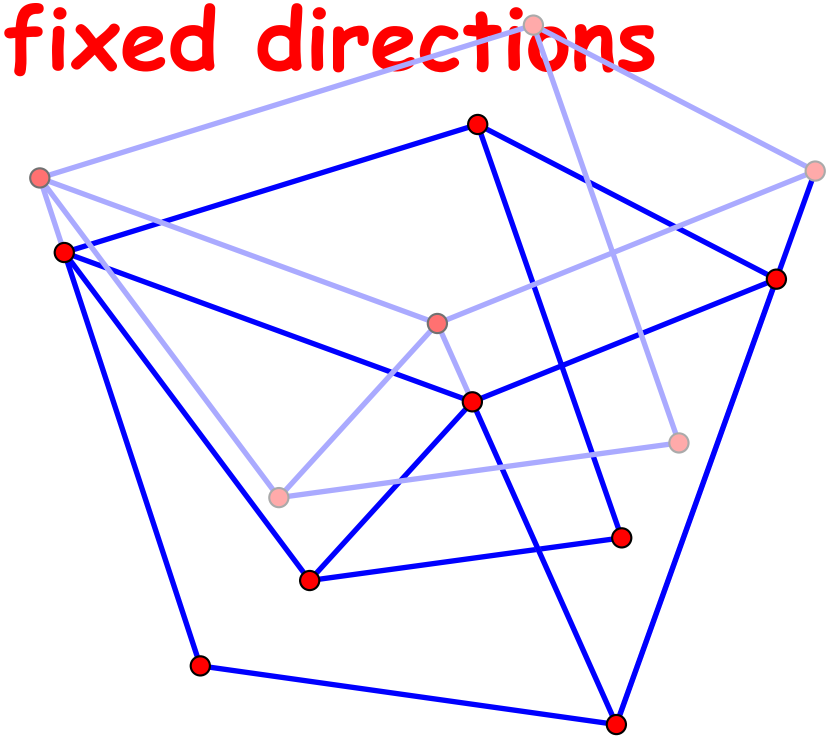
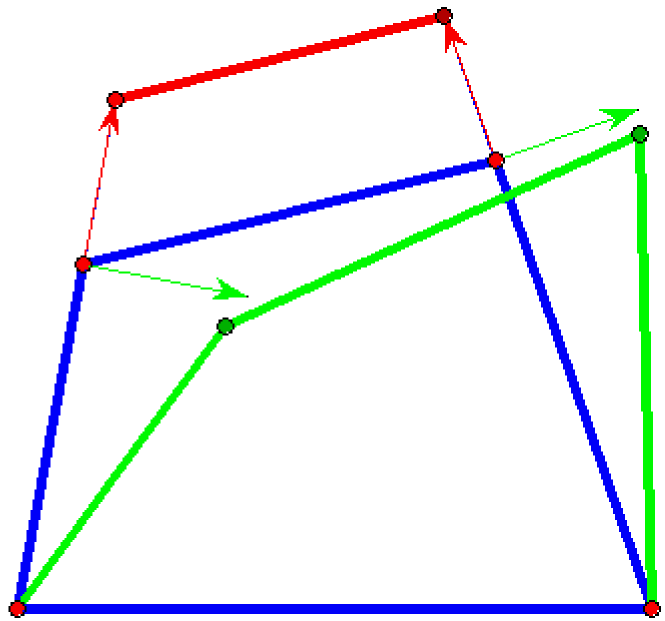


Laman mechanism (1dof flexible)

Non-trivial parallel redrawing

Background:

Relationship between motions in the two models



Orthogonal to each other

Reference: “folklore” from 19th century, see Whiteley, Matroids survey

Plan

- **Configuration spaces** of parallel redrawing Laman graphs and 1dof Laman mechanisms
- **Parallel Redrawing Sweep** for 1dof Laman mechanisms
- **Pointed pseudo-triangulation** mechanisms
- Further problems: **kinetic point sets** and **graphs, combinatorial sweeps**

Plan

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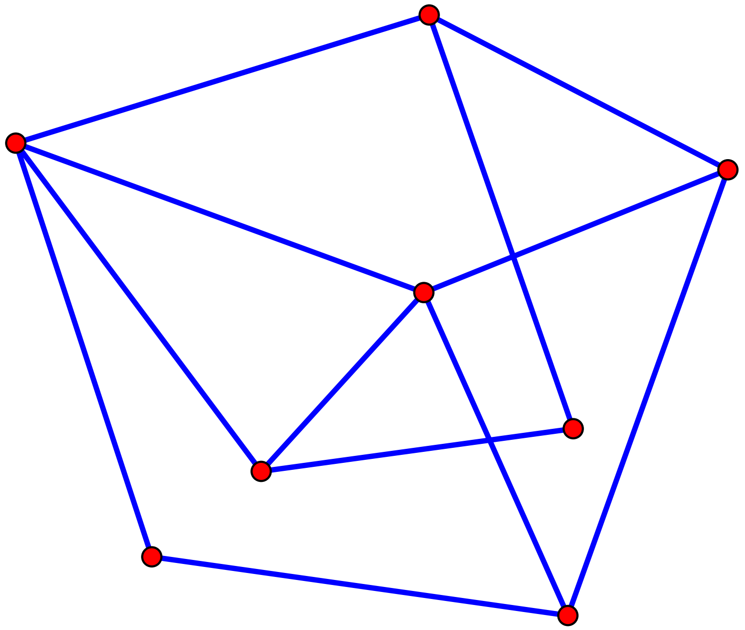
- **Parallel Redrawing Sweep** for 1dof Laman mechanisms
- **Pointed pseudo-triangulation** mechanisms
- Further problems: **kinetic point sets** and **graphs, combinatorial sweeps**

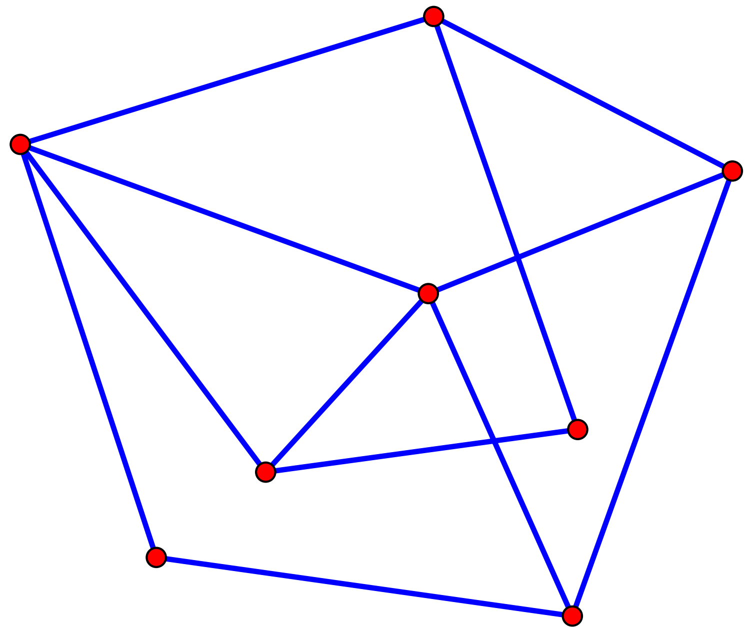
Direction networks and their parallel redrawings



Direction network (G,D):

- Graph G
- Set D of directions (slopes) for the edges

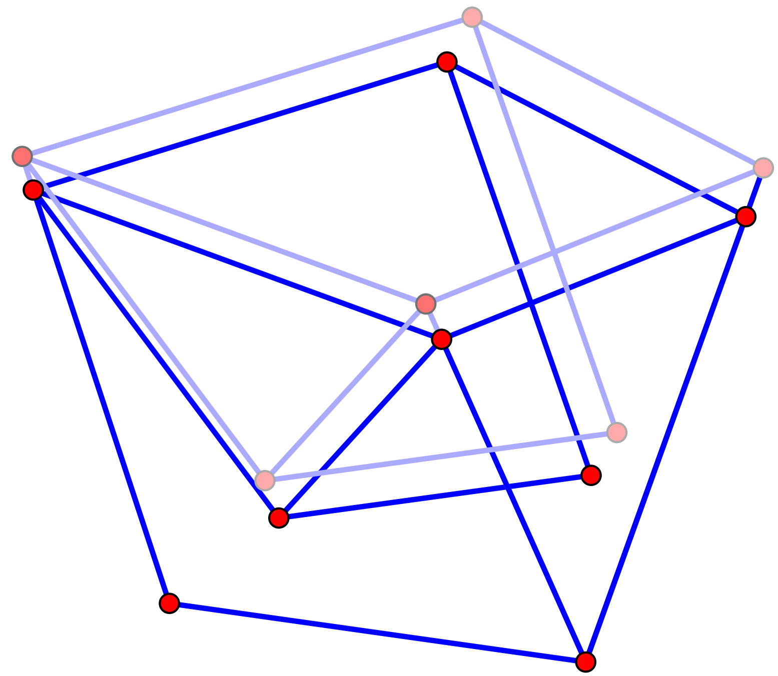
Direction networks and their parallel redrawings



Realization (embedding) of a direction network

- Mapping of vertices to points, edges to segments
- Consistent with given directions

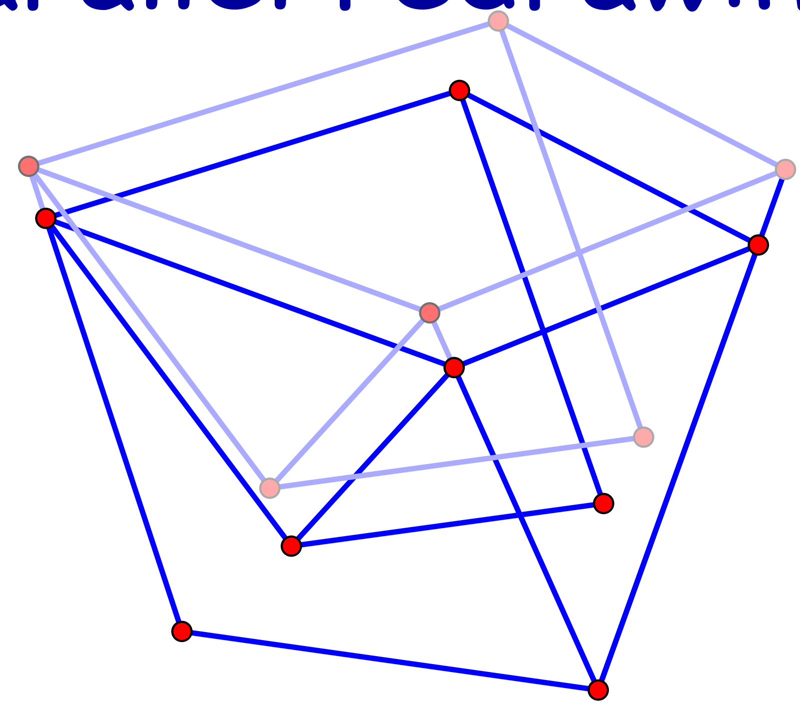
Direction networks and their parallel redrawings



Parallel redrawing of an embedded graph:

- Another realization of its underlying direction network
- Always can obtain similar ones by translation and rescaling
- Interesting: non-similar (non-trivial) parallel redrawings

Direction networks and their parallel redrawings



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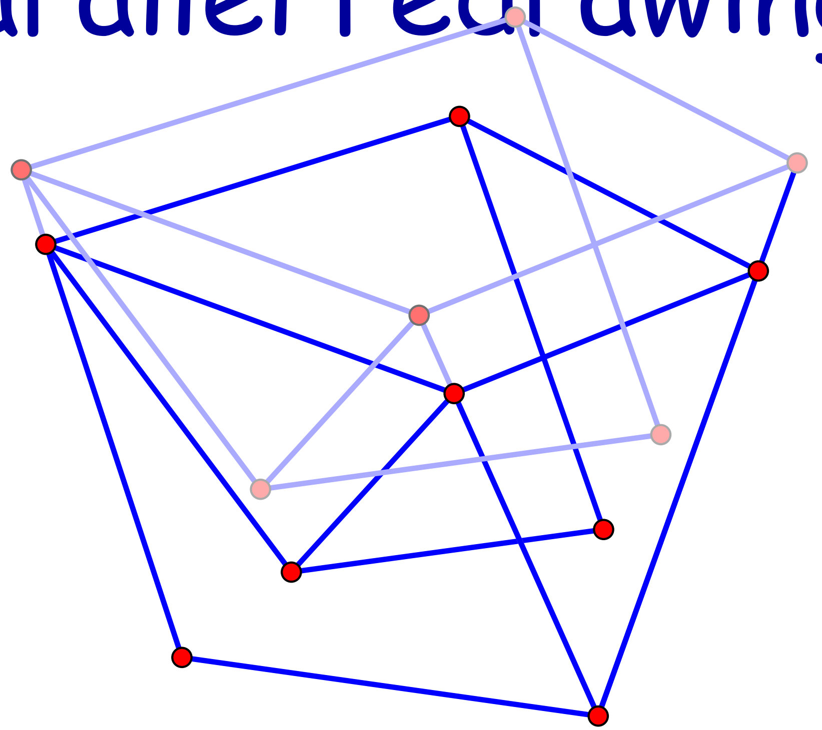

Parallel redrawing of an embedded graph:

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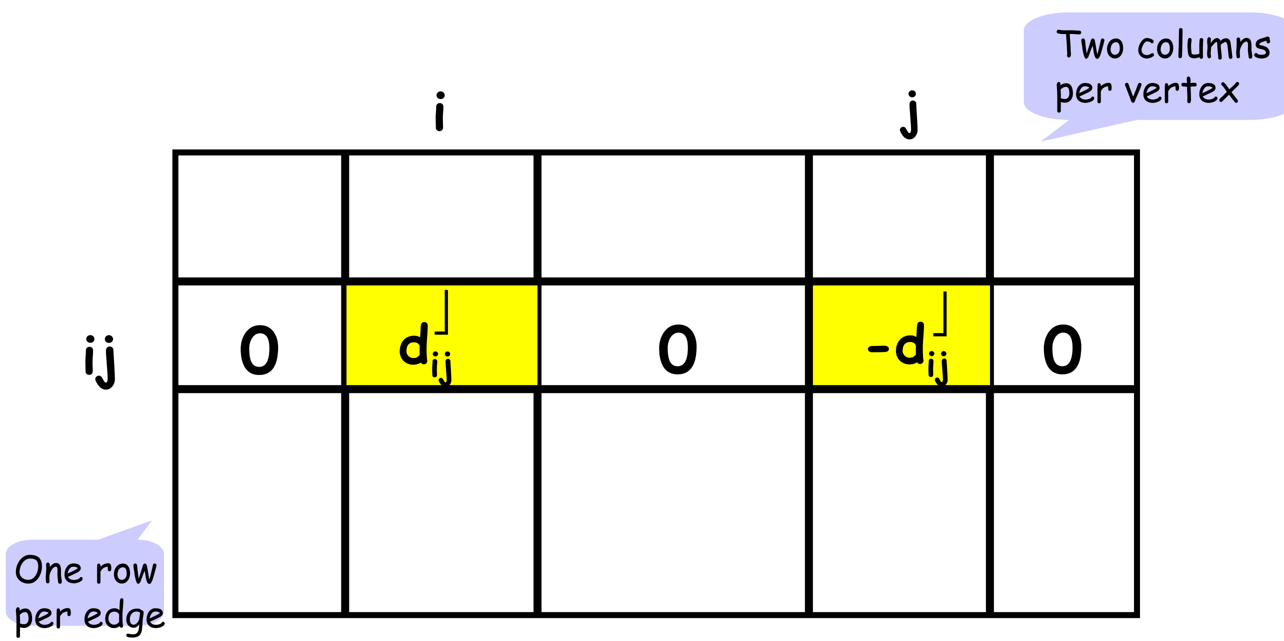
Realization space and Configuration space

- **Realization space:** set of all possible realizations of a direction network
- Linear subspace of R^{2n}: solutions of homogeneous linear system:
  - For all edges ij: $\langle p_i - p_j, d_{ij}^{\perp} \rangle = 0, \;\; \forall ij \in E$
  - Rp=0, where R is the “parallel redrawing matrix”



Realization space and Configuration space of a **1dof parallel redrawing Laman mechanism**

- **Realization space:** set of all possible realizations
- Linear subspace of R^{2n}: solutions of homogeneous linear system:
  - For all edges $\langle p_i - p_j, d_{ij}^{\perp} \rangle = 0, \quad \forall ij \in E$
  - Rp=0, where R is the "parallel redrawing matrix"
- Factor out translations: pin down a vertex, e.g. p_1=0. Still homogeneous, 2n-4 eqs, 2n-2 variables.
- **Configuration space: factor out scalings**
  - Projective view: factor out scalings (trivial parallel redrawings). Projective line in 2n-3 dim projective space.
  - Affine view: eliminate a "point at infinity". E.g. x_2=1. Affine line in Euclidean 2n-3 space
  - Oriented-projective view: factor by positive scalings. Oriented-projective line = great circle on 2n-3 sphere.

Rp=0 and Rp=b

- Rp=0 for a mechanism gives the realization space.
- Rp=b for a Laman graph, b all-but-one-zero vector, and one component acting as "time parameter":
  - Captures an affine part of the configuration space

Generic: R max rank

Rp=0 and Rp=b

- Rp=0 for a mechanism gives the realization space.
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Plan

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The Parallel Redrawing Sweep:

Visualizing the Configuration space of a 1dof Laman mechanism

Configuration space:

- **Projective view:** factor out scalings (trivial parallel redrawings). Projective line in 2n-3 dim projective space.
- **Affine view:** eliminate a "point at infinity" by pinning down one edge. Affine line in Euclidean 2n-3 space.

Lemma: Each point traces a linear trajectory: projection of config. space on the R^2 of the 2 coordinates of the point.

Lemma: Points move with constant velocities.

- **Oriented-projective view:** factor by positive scalings. Oriented-projective line = great circle on 2n-3 sphere. Points trace ellipses

The Parallel Redrawing Sweep:

Visualizing the Configuration space of a 1dof Laman mechanism

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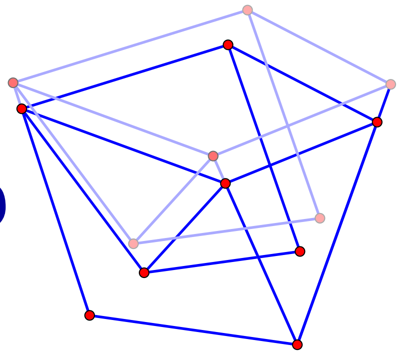
The parallel redrawing sweep

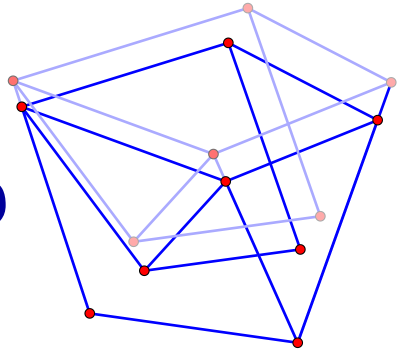


- Sweep of an affine part of the configuration space:
  - Determined by:
    - Choice of a rigid component (to be the "point at infinity"): "pinned down" edge.
    - Choice of an incident edge to "drive the sweep" (time parameter)
  - Sequence of collision events: rigid components collapse
- Understanding the sequence of collision events:
  - A rigid component "reverses" (rotates by 180 degrees, i.e. scales by a negative factor)

Lemma: At a collision event, the contraction of G on the collapsed edges is a Laman graph.

Lemma: The (constant) velocities are the "coordinates" of an "embedding" of the collapsed Laman graph on the r-component at "infinity".

The parallel redrawing sweep



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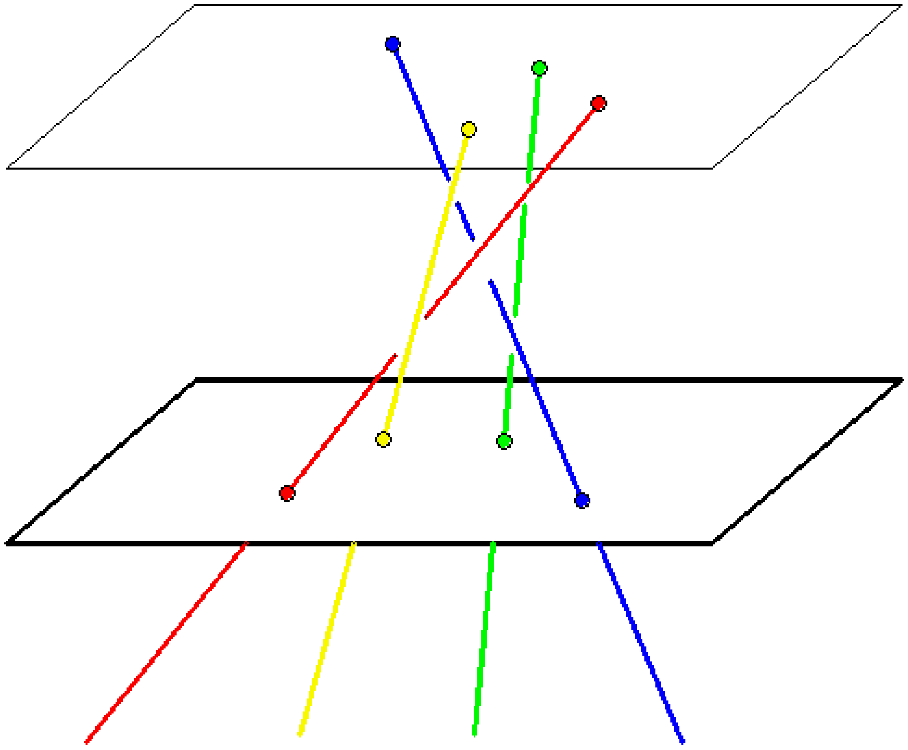
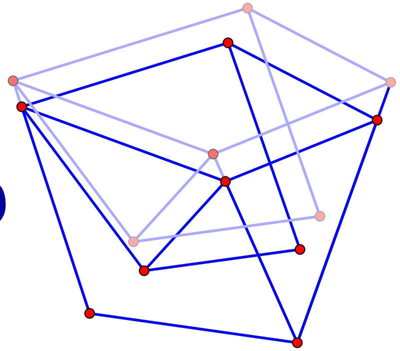
The parallel redrawing sweep: 3d (space-time) view



A plane sweep of a SPECIAL 3d line arrangement

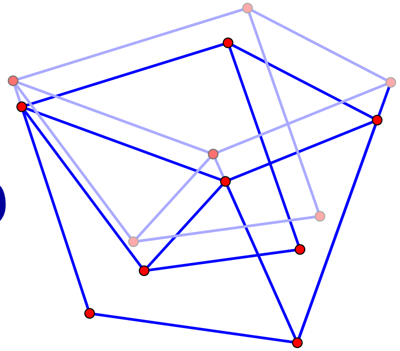
The parallel redrawing sweep



Algorithmic aspects:

- Computing the (combinatorial) events: rigid components of a Laman mechanism
- Predicting the next collision :
  - Linear algebra
  - Can it be done combinatorially?

The parallel redrawing sweep



Algorithmic aspects:

- Computing the (combinatorial) events: rigid components of a Laman mechanism
- Predicting the next collision :
  - Linear algebra
  - Can it be done combinatorially?

As in "topological sweep" versus "line sweep". Yes, for pseudo triangulation mechanisms

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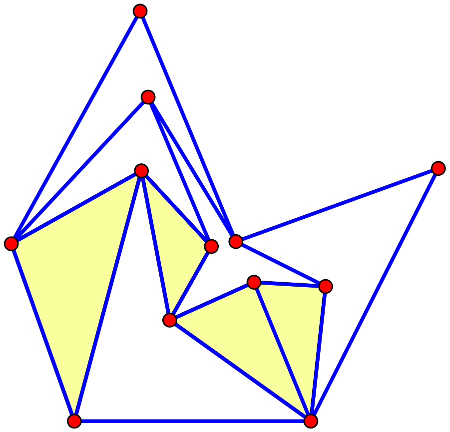
Parallel Redrawing Pointed pseudo-triangulation 1dof-mechanisms



- Pointed Pseudo-triangulation mechanisms (1dof, convex hull missing edge) are non-crossing throughout a parallel sweep

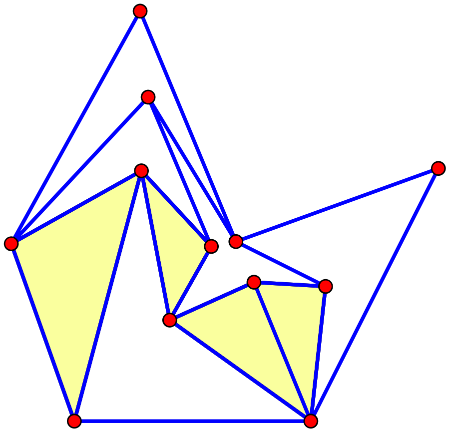
Proof:

- Use *expansiveness* property of pseudo-triangulation mechanisms:

$$\langle p_i - p_j, v_i - v_j \rangle = 0, \ \forall ij \in E$$
$$\langle p_i - p_j, v_i - v_j \rangle \geq 0, \ \forall ij \notin E$$

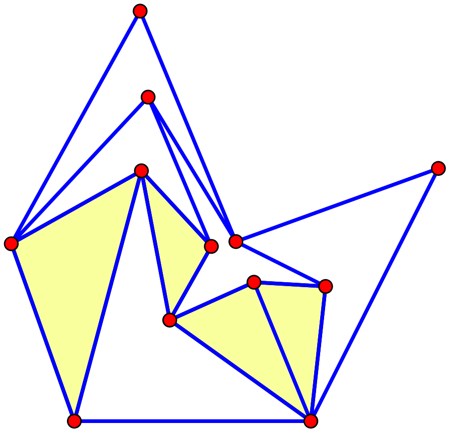
- Interpreted in parallel redrawing setting: **all are pseudo-triangulations**
- Look at oriented matroidal invariants maintained through the parallel sweep (partial unsigned hyperlines / combinatorial pseudo-triangulations): the **facial structure is maintained**

Parallel Redrawing Pointed pseudo-triangulation 1dof-mechanisms



- Lemma: $\sigma_{ij} = \langle p_i - p_j, d_{ij}^{\perp} \rangle$
  do not change during pr-sweep
- Corollary: signs don't change. Hence all positive (expansive). Hence it is a pt-mechanism.
- Lemma: next event only among the "flippable" r-components. Must be extreme at incident joints, and incident components oriented the same way. Property maintained after the event ("flip").
- Lemma: Planar face structure doesn't change. Only combinatorial pseudo-triangulation.
- Obs: Captures partial oriented matroid information (signed hyperlines) of embedded graph.

Parallel Redrawing Pointed pseudo-triangulation 1dof-mechanisms



- Pointed Pseudo-triangulation mechanisms (1dof, convex hull missing edge) are non-crossing throughout a parallel sweep
- Replacing a rigid component by another non-crossing graph (on the same kinetic point set) is again a kinetic non-crossing graph

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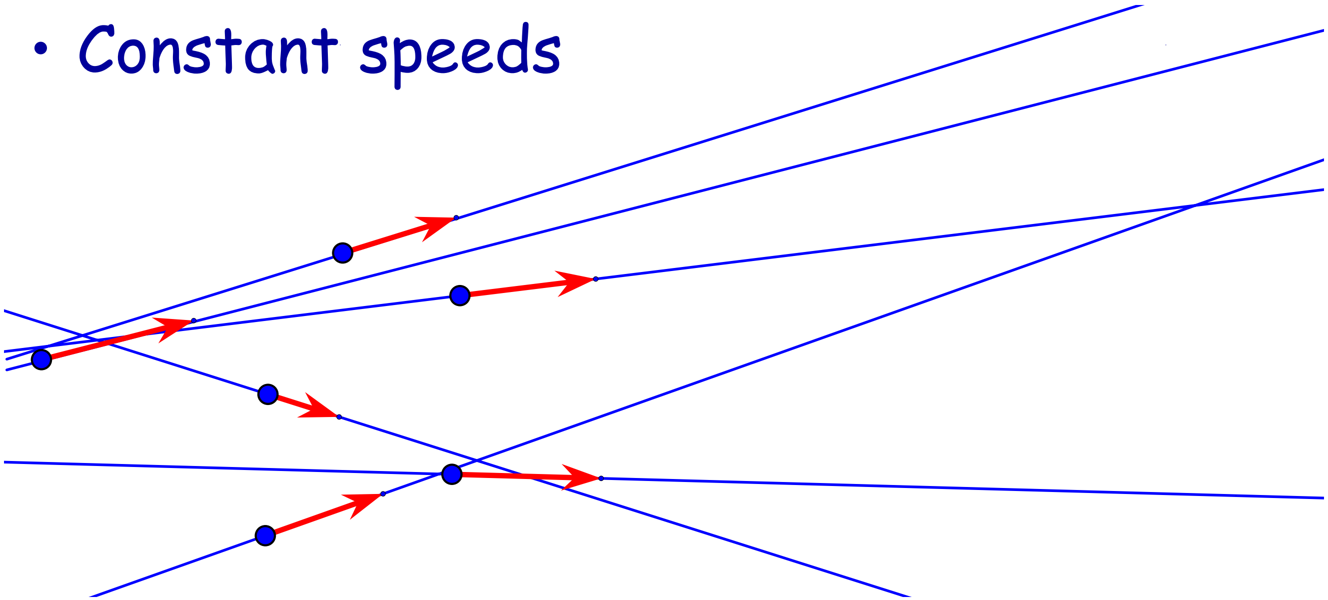
Study collisions in:

Kinetic Point Sets

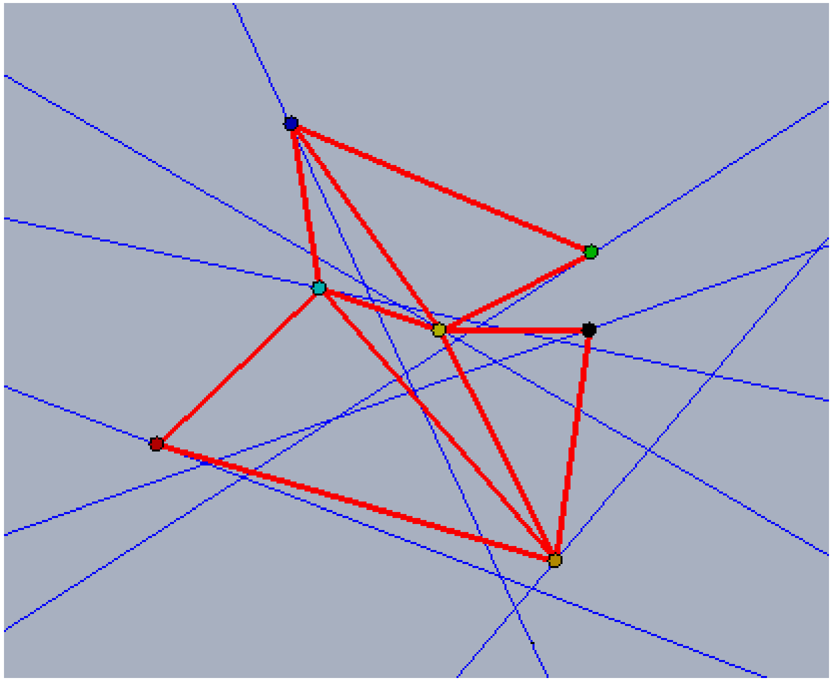
- Points in continuous motion
- Moving with constant velocities
    - Linear trajectories
    - Constant speeds



Study crossings in:

Kinetic Graphs

- Graphs drawn (embedded) on kinetic point sets



Study:

Combinatorial (Topological) Parallel Redrawing Sweep

- For pointed combinatorial pseudo-triangulation mechanisms
- Next event predicted combinatorially
- Is every combinatorial sequence realizable?

Questions?

http://cs.smith.edu/~streinu/Research/KineticPT