Folding & Unfolding: Folding Polygons to Convex Polyhedra

Joseph O'Rourke
Smith College

Folding and Unfolding Talks

Linkage folding	Tuesday	Erik Demaine
Paper folding	Wednesday	Erik Demaine
Folding polygons into convex polyhedra	**Saturday_1**	**Joe O'Rourke**
Unfolding polyhedra	Saturday_2	Joe O'Rourke

Outline

- Reconstruction of Convex Polyhedra
  - Cauchy to Sabitov (to an Open Problem)
- Folding Polygons
  - Algorithms
  - Examples
  - Questions

Outline$_1$

- Reconstruction of Convex Polyhedra
  - Cauchy to Sabitov (to an Open Problem)
    - Cauchy's Rigidity Theorem
    - Aleksandrov's Theorem
    - Sabitov's Algorithm

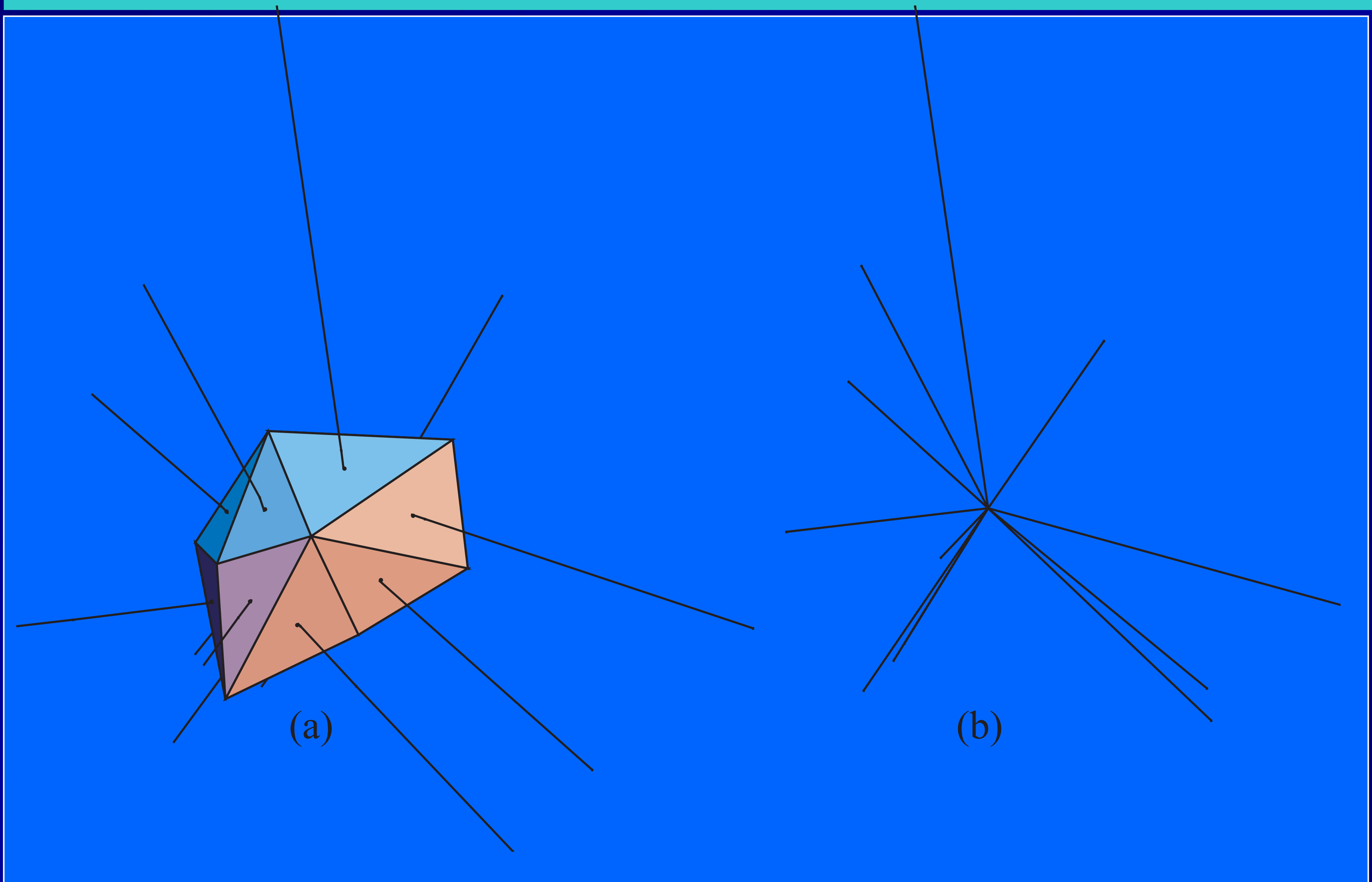
Outline$_2$

- Folding Polygons
  - Algorithms
    - Edge-to-Edge Foldings
    - Gluing Trees; exponential lower bound
    - Gluing Algorithm
  - Examples
    - Foldings of the Latin Cross
    - Foldings of the Square
  - Questions
    - Transforming shapes?

Reconstruction of Convex Polyhedra

- graph — Steinitz's Theorem
- face angles
- edge lengths
- face areas }
- face normals } Minkowski's Theorem
- dihedral angles
- inscribed/circumscribed

Minkowski's Theorem



Reconstruction of Convex Polyhedra

- graph
- face angles
- edge lengths

} Cauchy's Theorem

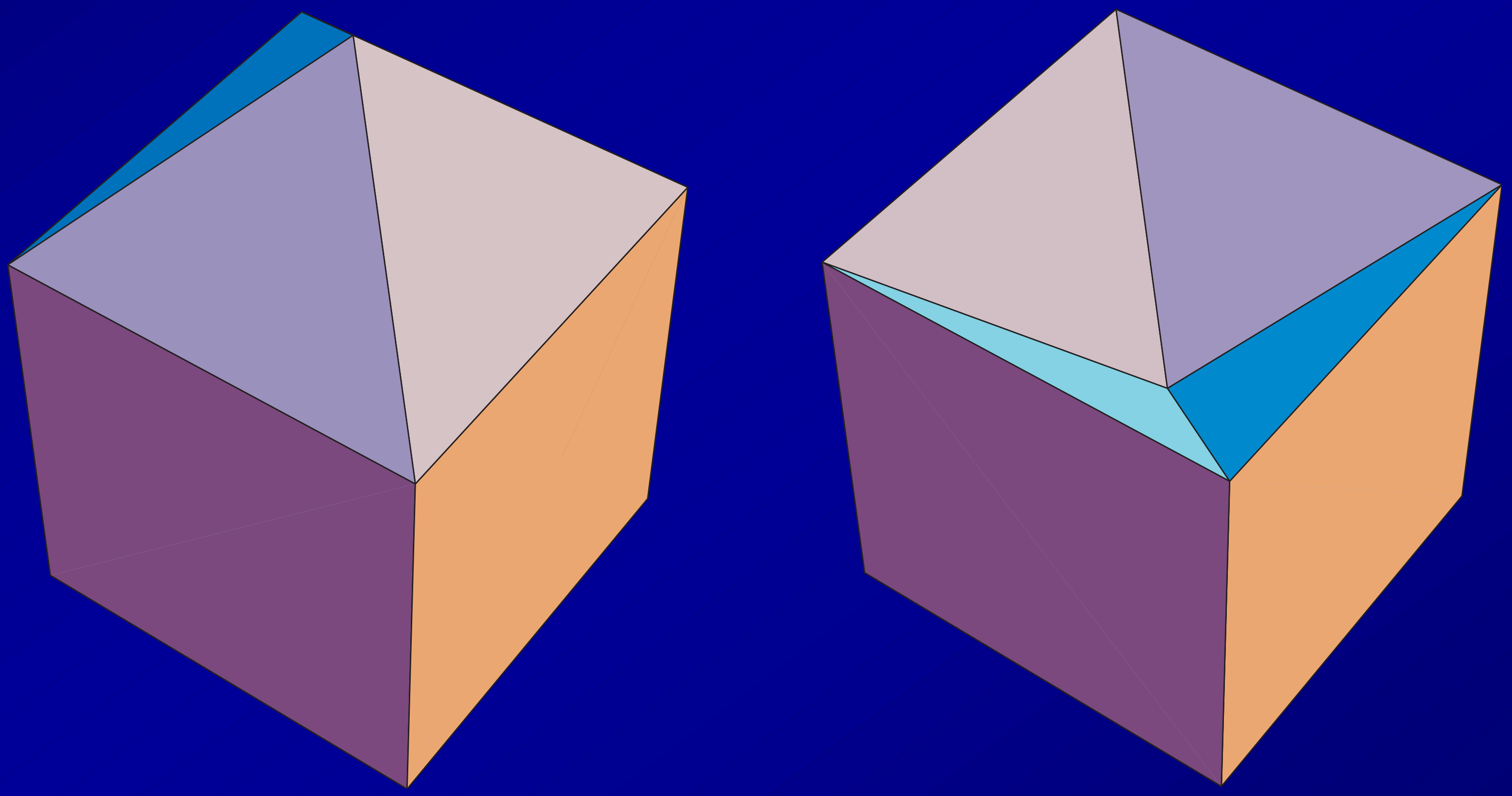
- face areas
- face normals
- dihedral angles
- inscribed/circumscribed

Cauchy's Rigidity Theorem

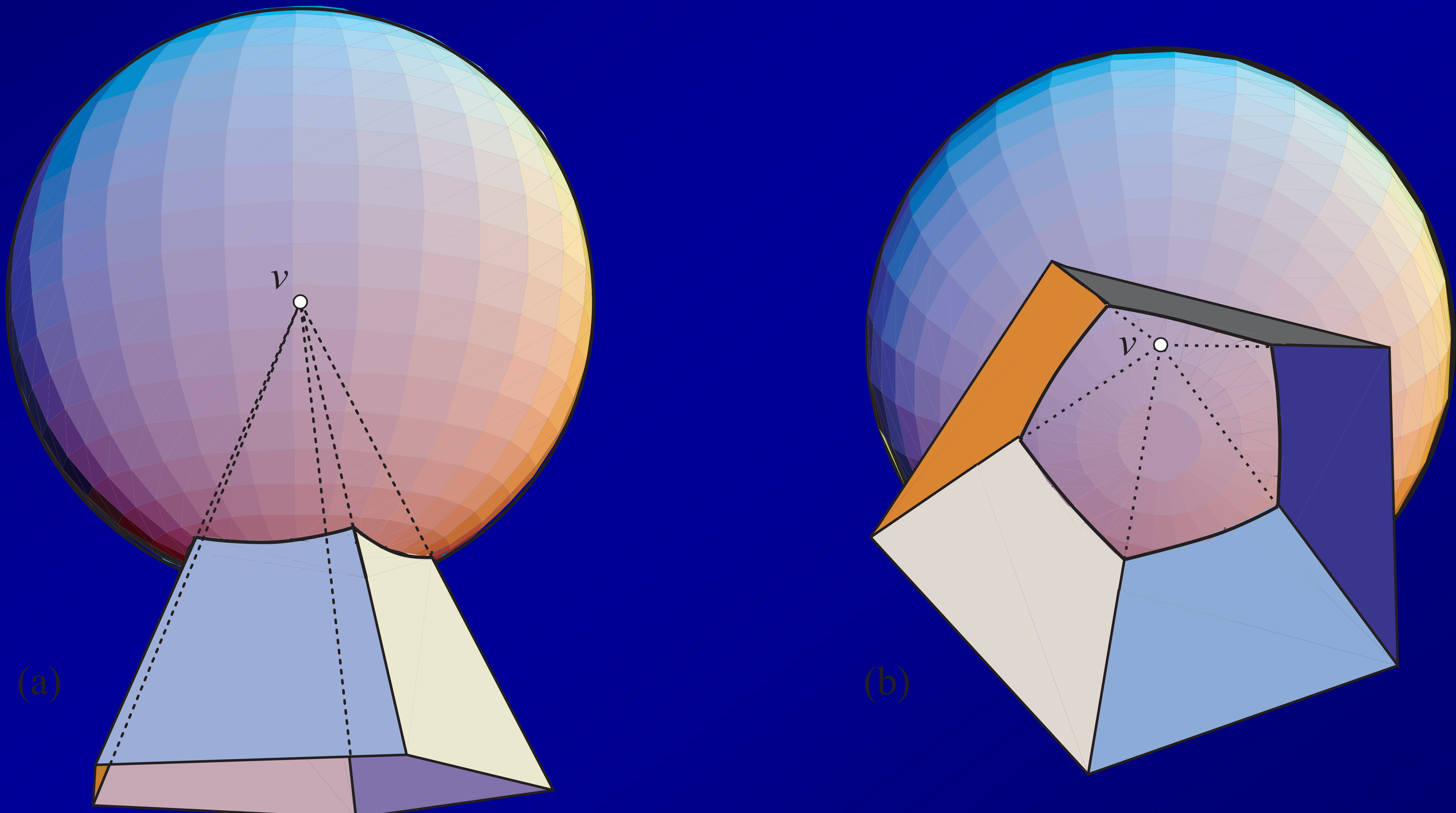
- If two closed, convex polyhedra are combinatorially equivalent, with corresponding faces congruent, then the polyhedra are congruent;
- in particular, the dihedral angles at each edge are the same.

Global rigidity == unique realization

Same facial structure, noncongruent polyhedra



Spherical polygon

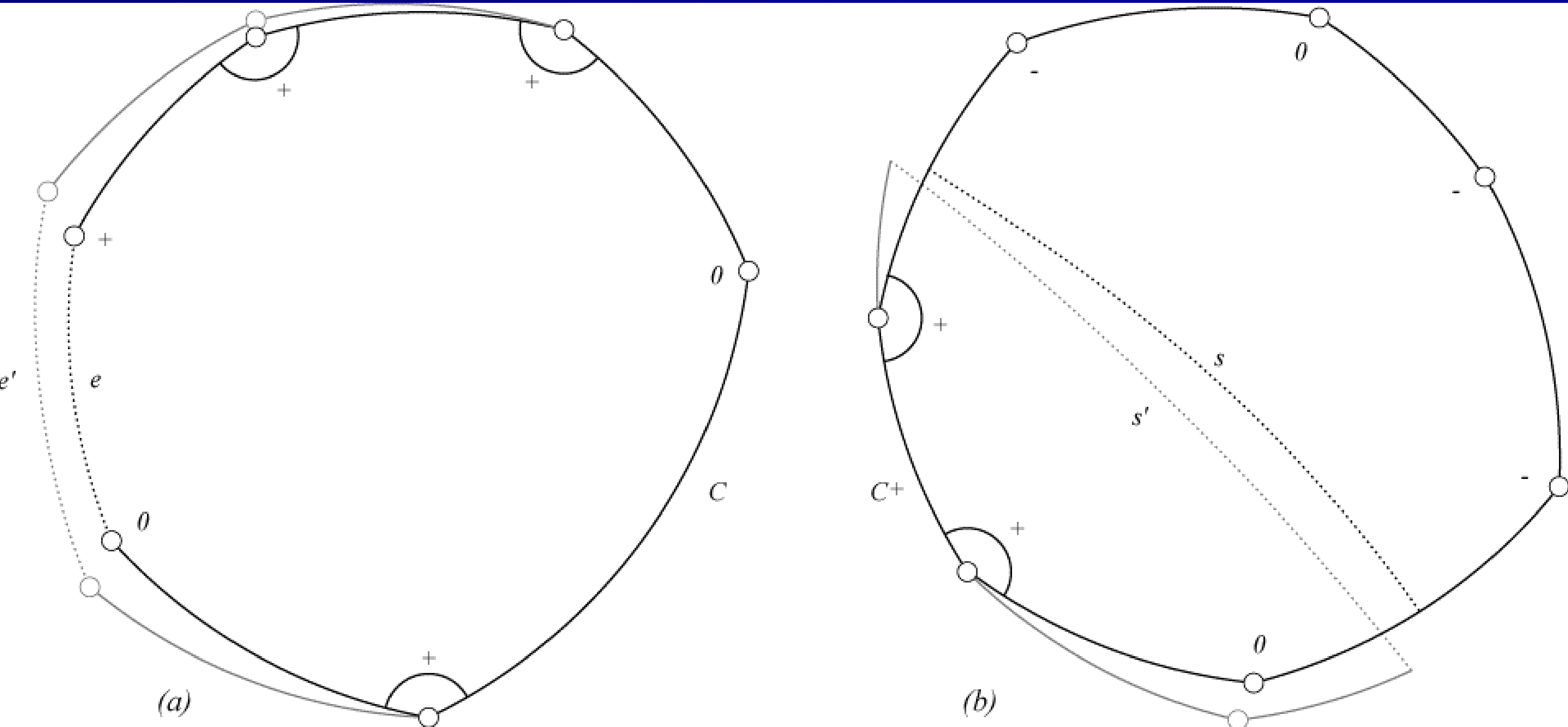


Sign Labels: {+, -, 0}

- Compare spherical polygons Q to Q′
- Mark vertices according to dihedral angles: {+,-,0}.

Lemma: The total number of alternations in sign around the boundary of Q is ≥ 4.

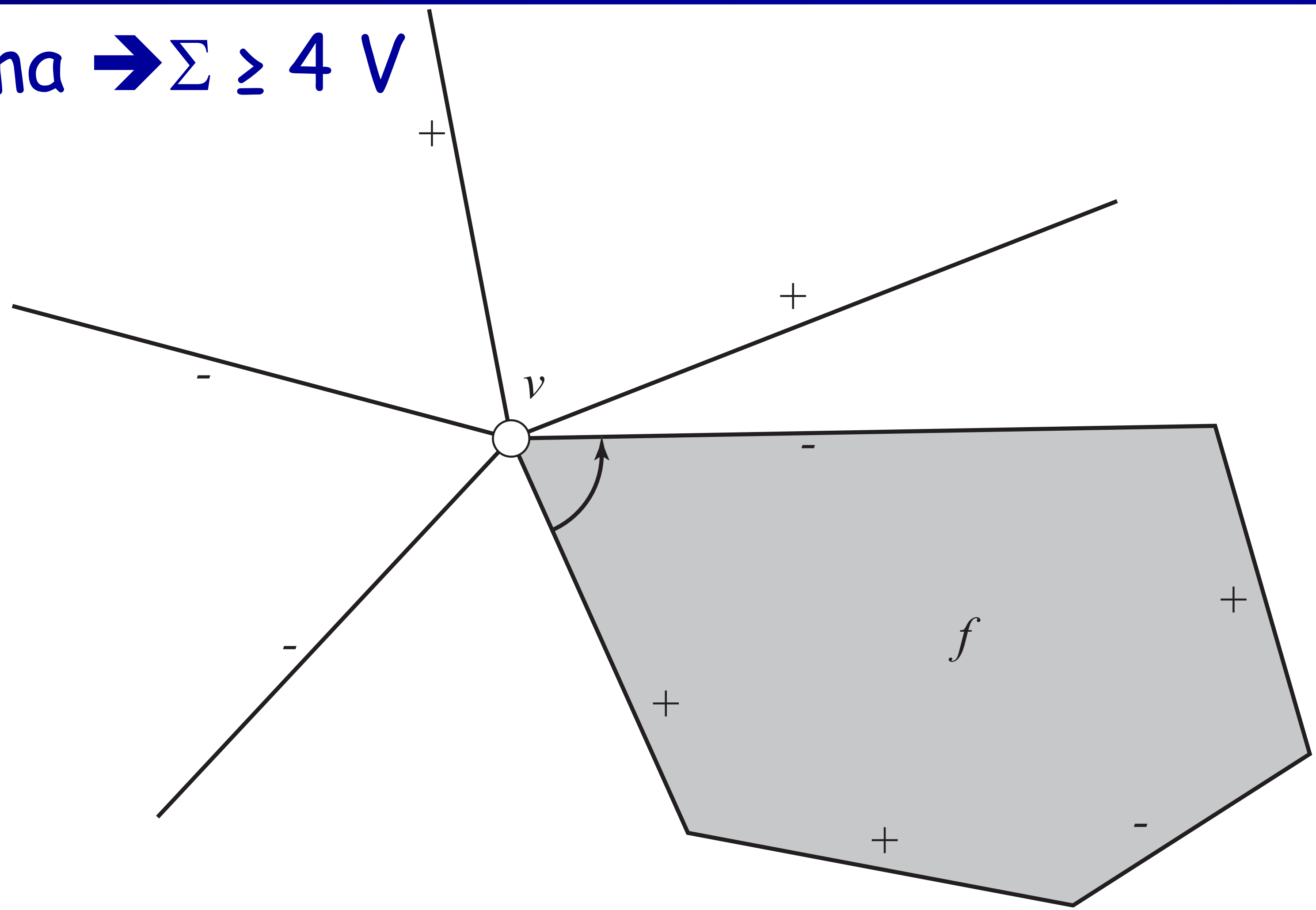
The spherical polygon opens.



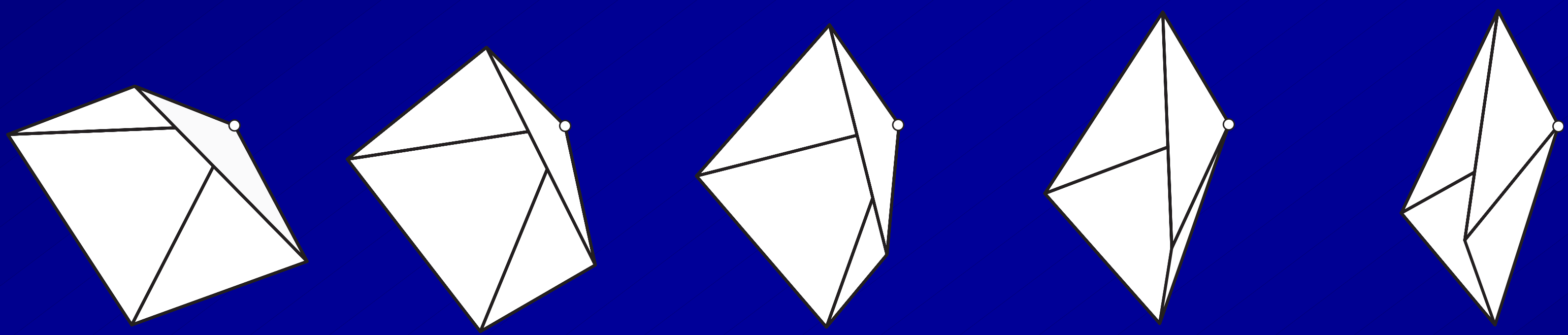
(a) Zero sign alternations; (b) Two sign alts.

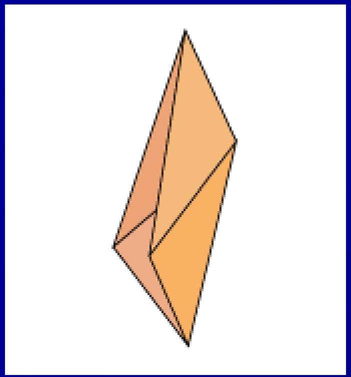
Sign changes → Euler Theorem Contradiction

Lemma → $\Sigma \geq 4V$

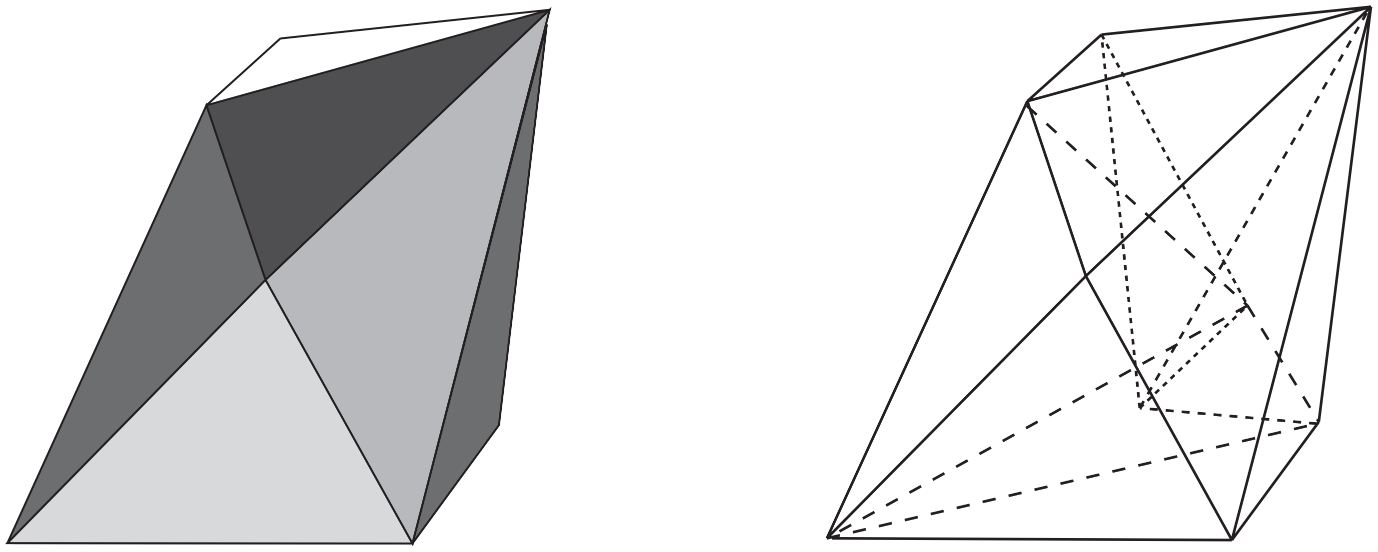


Flexing top of regular octahedron





Steffen's flexible polyhedron



14 triangles, 9 vertices

http://www.mathematik.com/Steffen/

The Bellow's Conjecture

- Polyhedra can bend but not breathe [Mackenzie 98]
- Settled in 1997 by Robert Connelly, Idzhad Sabitov, and Anke Walz
- Heron's formula for area of a triangle:

  $A^2 = s(s-a)(s-b)(s-c)$
- Francesca's formula for the volume of a tetrahedron
- Sabitov: volume of a polyhedron is a polynomial in the edge lengths.

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    - Cauchy's Rigidity Theorem
    - Aleksandrov's Theorem
    - Sabitov's Algorithm

Aleksandrov's Theorem (1941)

- “For every convex polyhedral metric, there exists a unique polyhedron (up to a translation or a translation with a symmetry) realizing this metric."

Pogorelov's version (1973)

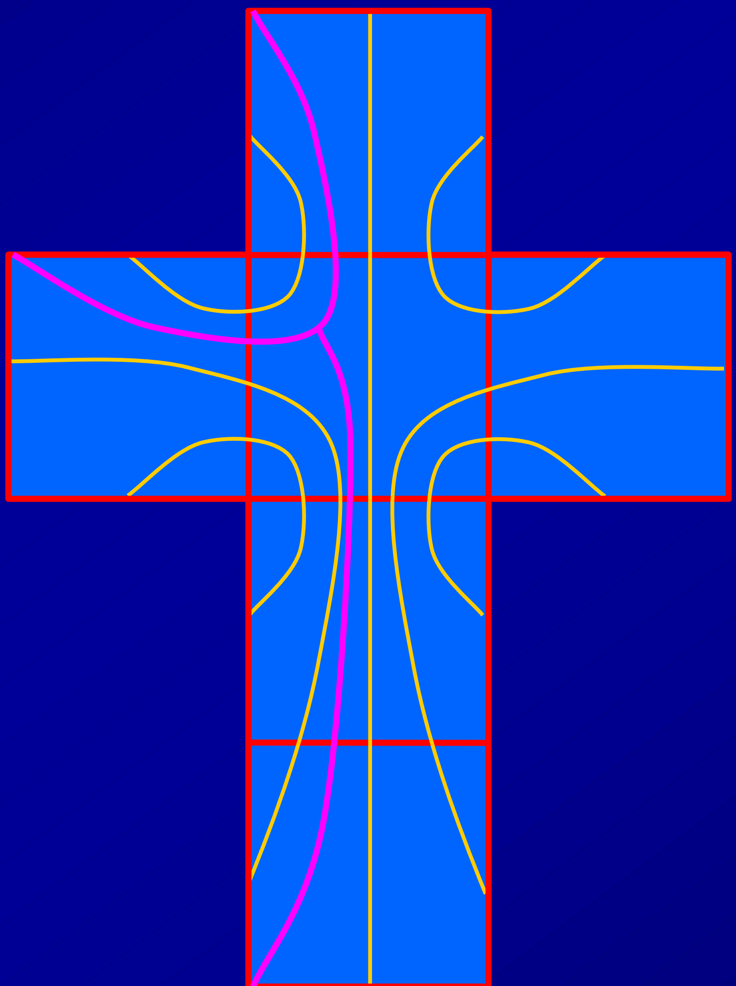
- "For every convex polyhedral metric, there exists a unique polyhedron (up to a translation or a translation with a symmetry) realizing this metric."
- "Any convex polyhedral metric given ... on a manifold homeomorphic to a sphere is realizable as a closed convex polyhedron (possibly degenerating into a doubly covered plane polygon)."

Alexandrov Gluing (of polygons)

- Uses up the perimeter of all the polygons with boundary matches:
  - No gaps.
  - No paper overlap.
  - Several points may glue together.
- At most 2π angle at any glued point.
- Homeomorphic to a sphere.

Aleksandrov's Theorem $\Rightarrow$ unique "polyhedron"

Folding the Latin Cross



Cauchy vs. Aleksandrov

- Cauchy: uniqueness
- Aleksandrov: existence and uniqueness

- Cauchy: faces and edges specified
- Aleksandrov: gluing unrelated to creases

Uniqueness

- Cauchy: combinatorial equivalence + congruent faces => congruent
- Aleksandrov: "Two isometric polyhedra are equivalent"
- The sphere is rigid [Minding]
- The sphere is unique given its metric [Liebmann, Minkowski]
- Closed regular surfaces are rigid [Liebmann, Blaschke, Weyl]
- Uniqueness of these w/in certain class [Cohn-Vossen]
- …
- "Isometric closed convex surfaces are congruent" [Pogorelov 73]

Alexandrov Existence$_1$

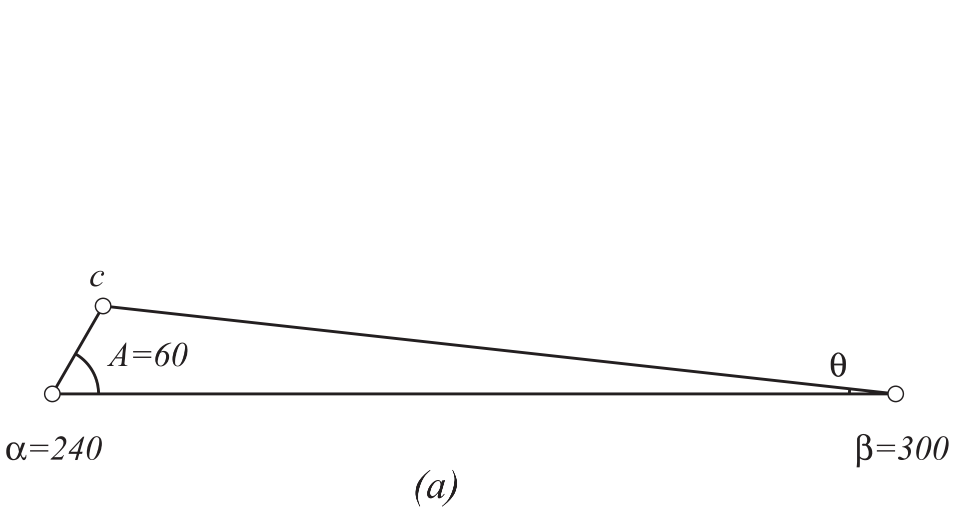
- Induction on the number of vertices n of the metric:
  - from realization of n-1 vertex metric to n vertex metric
  - by continuous deformation of metrics
  - tracked by polyhedral realizations

Existence$_2$

curvature = $2\pi - \Sigma$ angs

- There are two vertices a and b with curvature less than π.
- Connect by shortest path γ.
- Cut manifold along γ and insert double Δ that leaves curvature unchanged.
- Adjust shape of Δ until a and b both disappear: n-1 vertices.
- Realize by induction, introduce nearby pseudovertex, track sufficiently close metrics.

(a) a flattened; (b) b flattened.

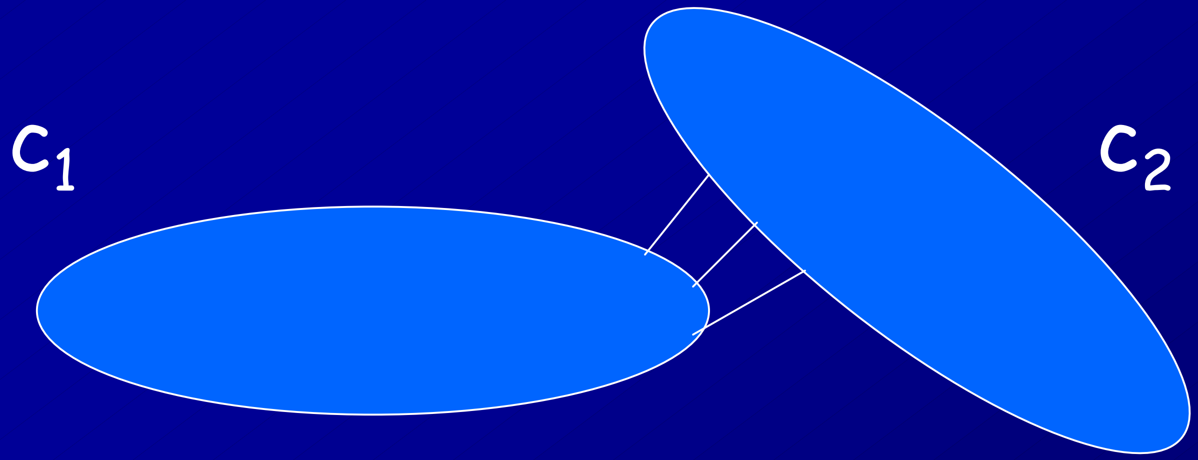


D-Forms

Smooth closed convex curves of same perimeter.

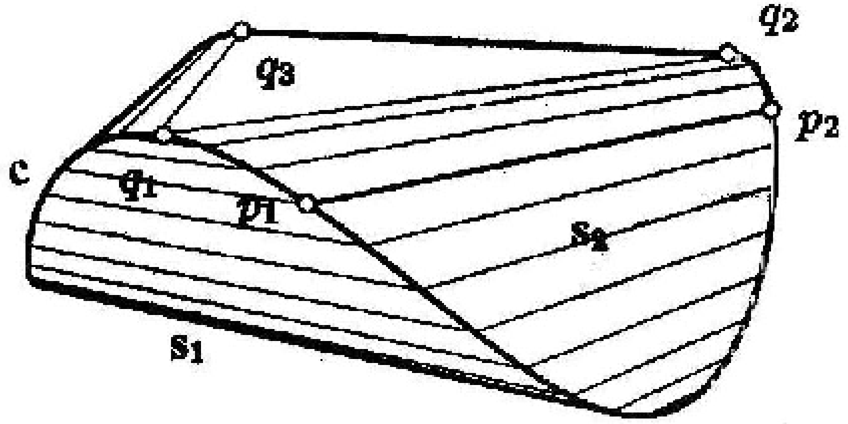
Glue perimeters together.

→ D-form



Helmut Pottmann and Johannes Wallner. *Computational Line Geometry*. Springer-Verlag, 2001.

Fig 6.49, p.401

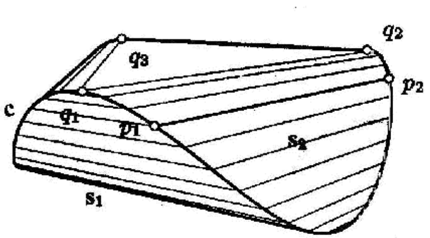


Pottmann & Wallner

- When is a D-form is the convex hull of a space curve? Always

- When is it free of creases?

Always



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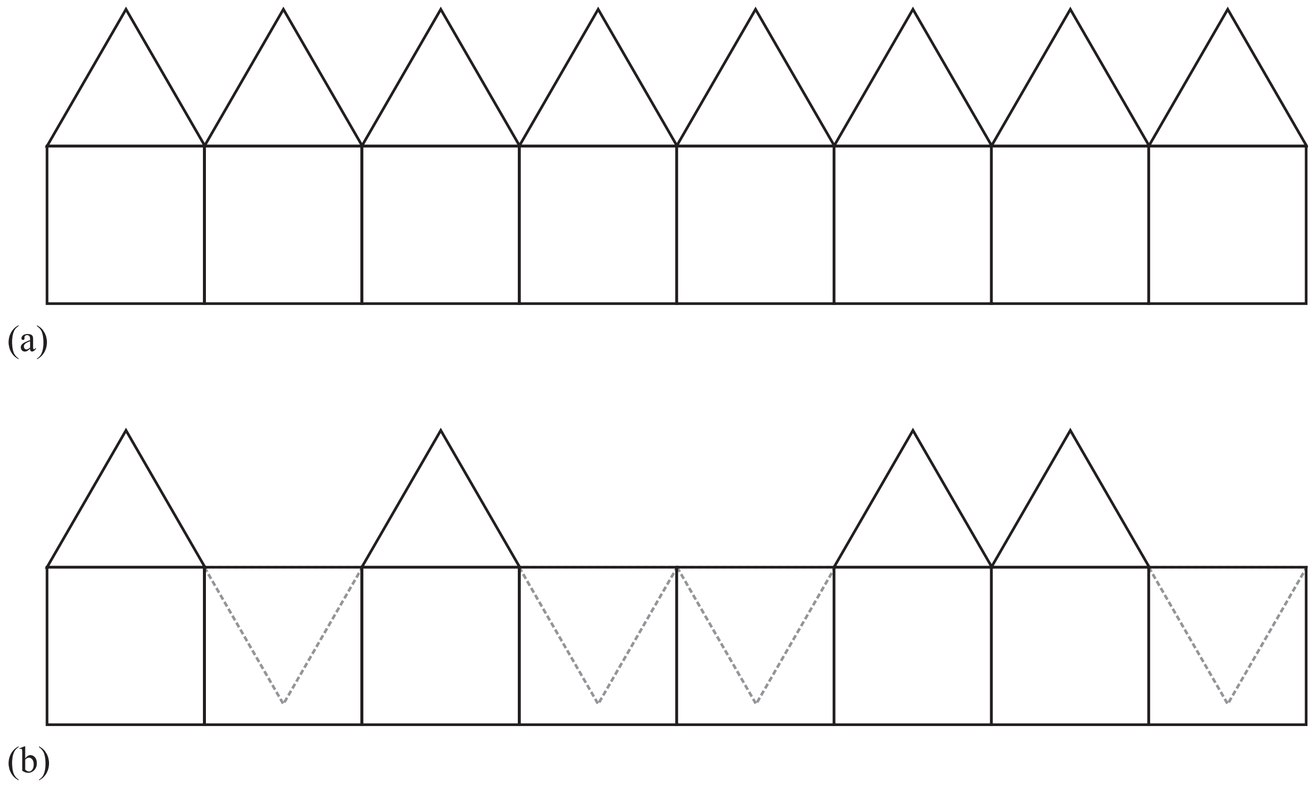
Sabitov's Algorithm

- Given edge lengths of triangulated convex polyhedron,
- computes vertex coordinates
- in time exponential in the number of vertices.

Sabitov Volume Polynomial

- $V^{2N} + a_1(l)V^{2(N-1)} + a_2(l)V^{2(N-2)} + \ldots$
  $+ a_N(l)V^0 = 0$
  - Tetrahedron: $V^2 + a_1(l) = 0$
  - l = vector of six edge lengths
  - $a_1(l) = \sum \delta_{ijk} (l_i)^2(l_j)^2(l_k)^2 /144$
  - Francesca's formula
- Volume of polyhedron is root of polynomial

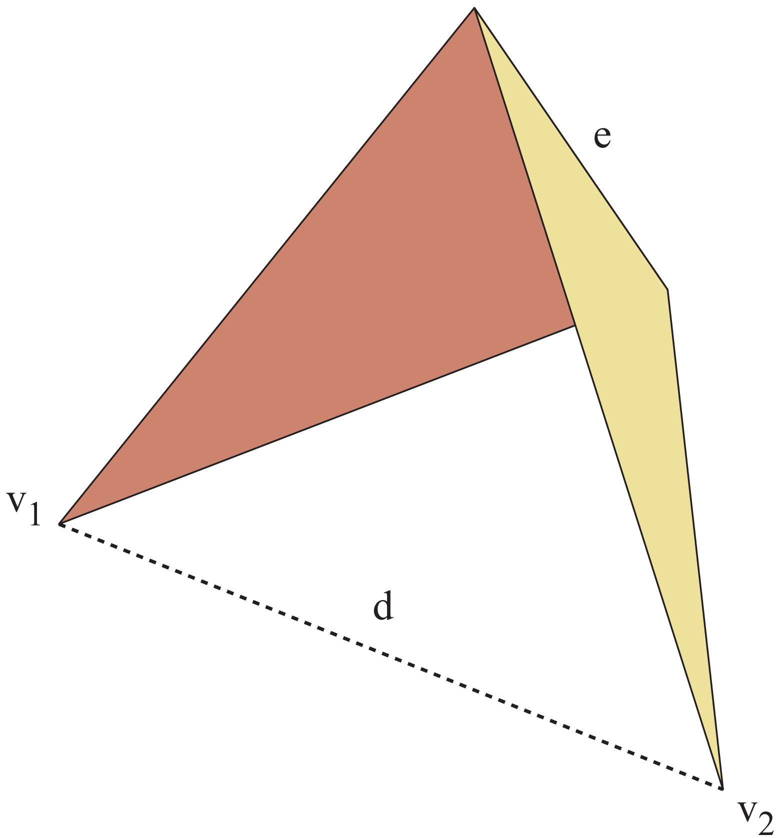
2^N possible roots



Generalized Polyhedra

- Polynomial represents volume of generalized polyhedra
  - any simplicial 2-complex homeomorphic to an orientable manifold of genus ≥ 0
  - mapped to R^3 by continuous function linear on each simplex.
- Need not be embeddable: surface can self-intersect.

Sabitov Proof$_1$



Sabitov Proof$_2$

- $vol(P) = vol(P') - \delta\, vol(T)$, $\delta = \pm 1$
- ... [many steps] ...
- polynomial = 0
- unknowns:
  - edge lengths vector l
  - $V = vol(P)$
  - unknown diagonal d

Sabitov Proof_3

- unknowns:
  - edge lengths l [given]
  - V = vol(P) ["known" from volume polynomial]
  - unknown diagonal d
- Try all roots for V, all roots for d: candidates for length of d & dihedral at e.
- Repeat for all e.
- Check the implied dihedral angles for the convex polyhedron.

Open: Practical Algorithm for Cauchy Rigidty

Find either

- a polynomial-time algorithm,
- or even a numerical approximation procedure,

that takes as

- input the combinatorial structure and edge lengths of a triangulated convex polyhedron, and
- outputs coordinates for its vertices.

Outline$_2$

- Folding Polygons
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    - Edge-to-Edge Foldings
    - Gluing Trees; exponential lower bound
    - Gluing Algorithm
  - Examples
    - Foldings of the Latin Cross
    - Foldings of the Square
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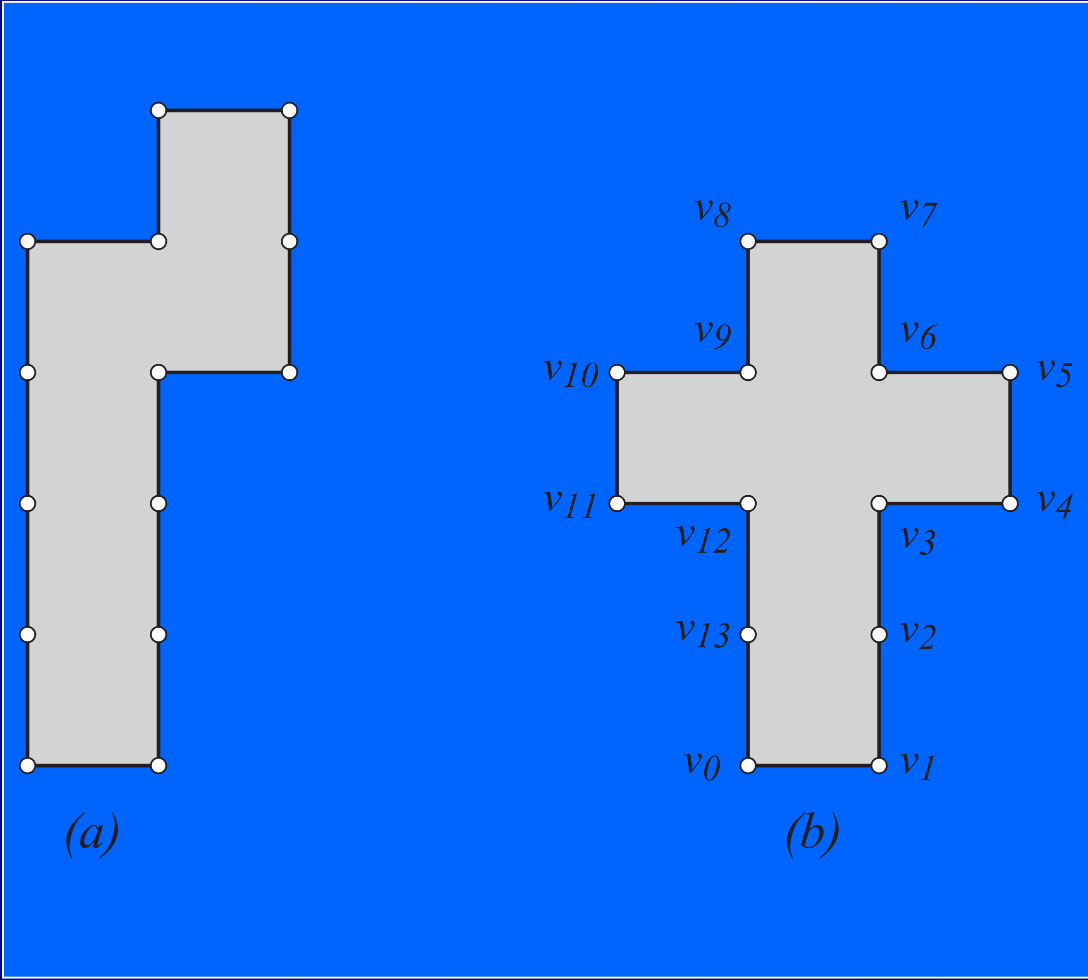
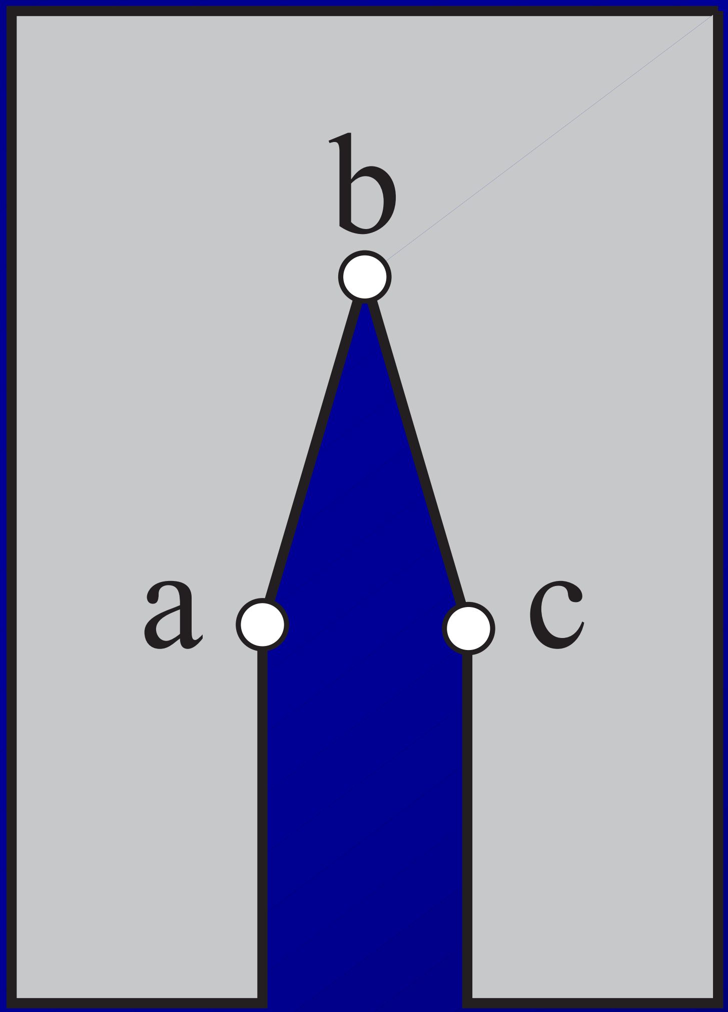
Folding Polygons to Convex Polyhedra

- When can a polygon fold to a polyhedron?
  - "Fold" = close up perimeter, no overlap, no gap :
- When does a polygon have an Aleksandrov gluing?



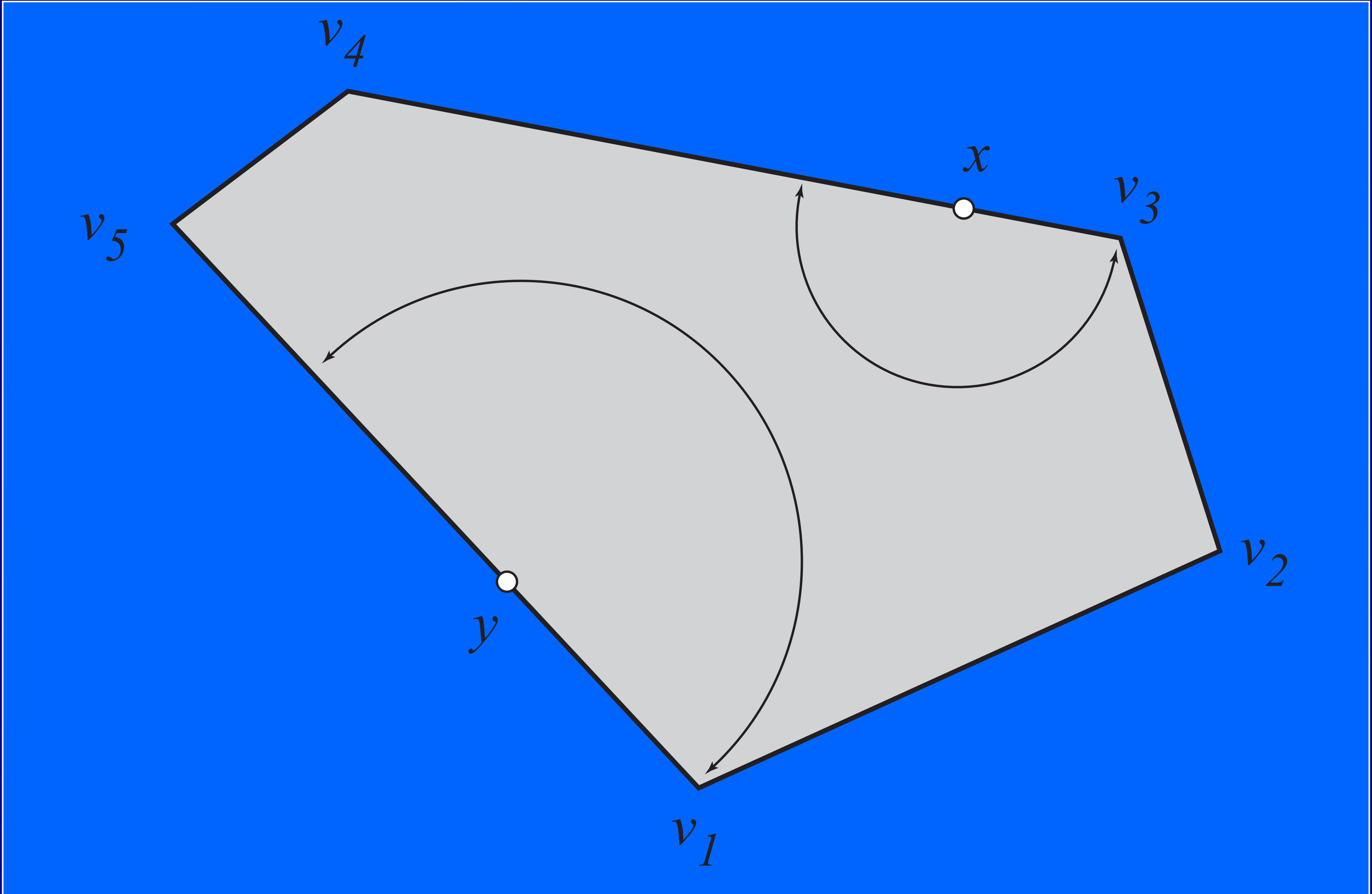
Unfoldable Polygon



Foldability is "rare"

Lemma: The probability that a random polygon of n vertices can fold to a polytope approaches 0 as $n \to 1$.

Perimeter Halving

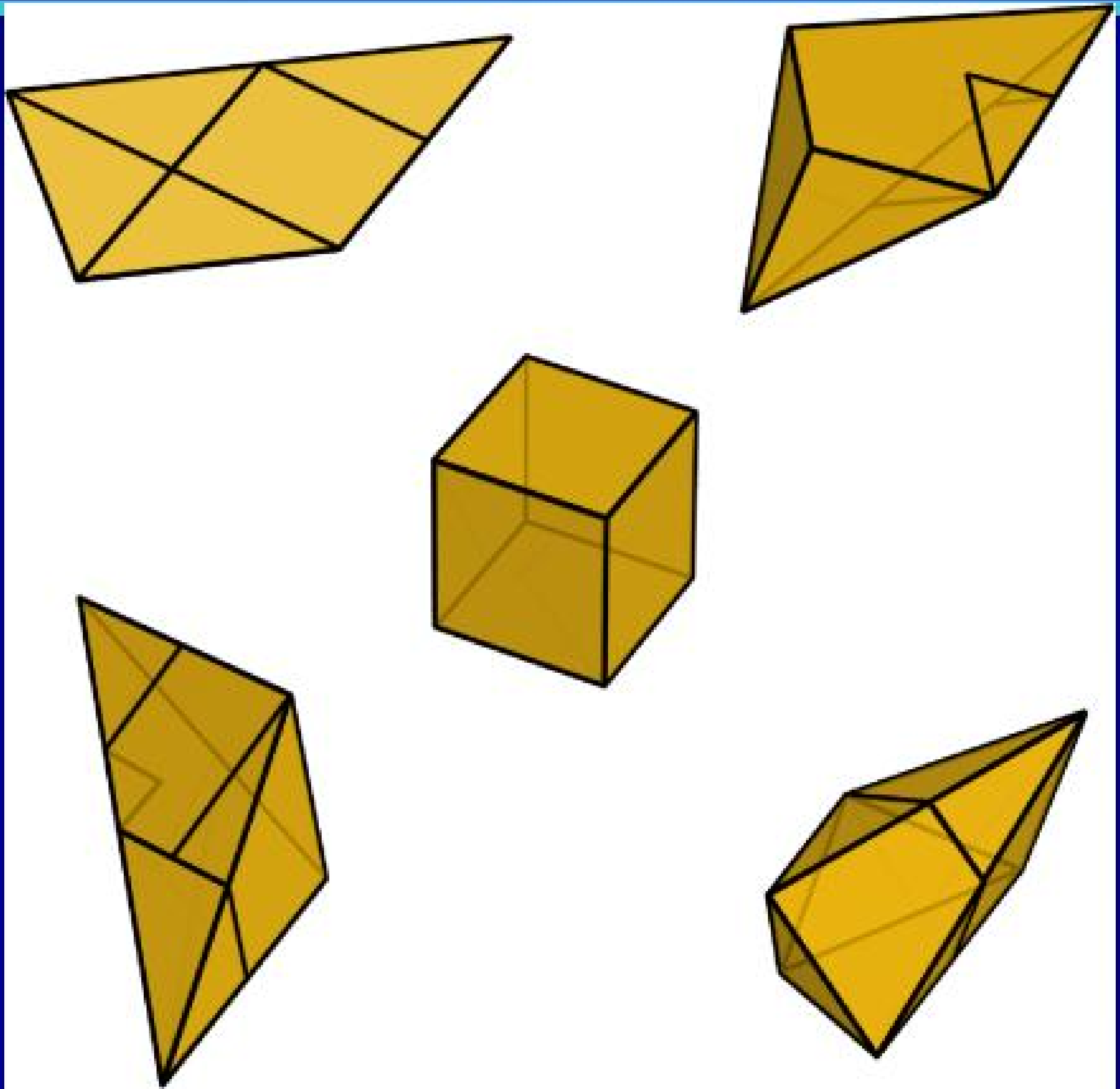


Edge-to-Edge Gluings

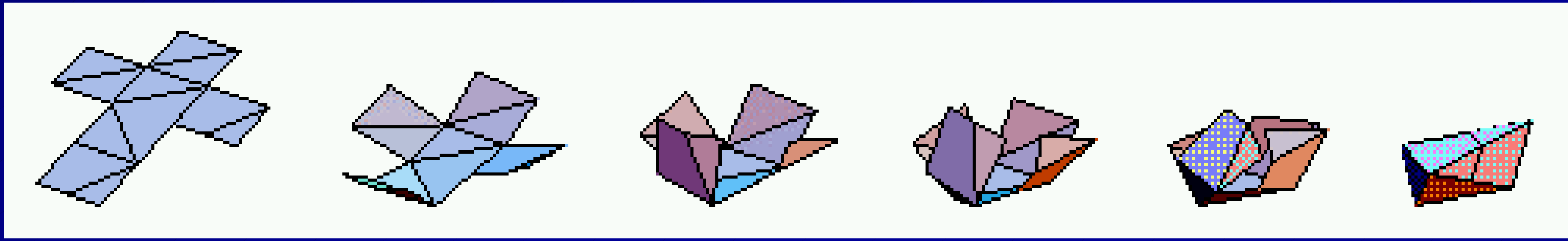
- Restricts gluing of whole edges to whole edges.

[Lubiw & O'Rourke, 1996]

New Re-foldings of the Cube

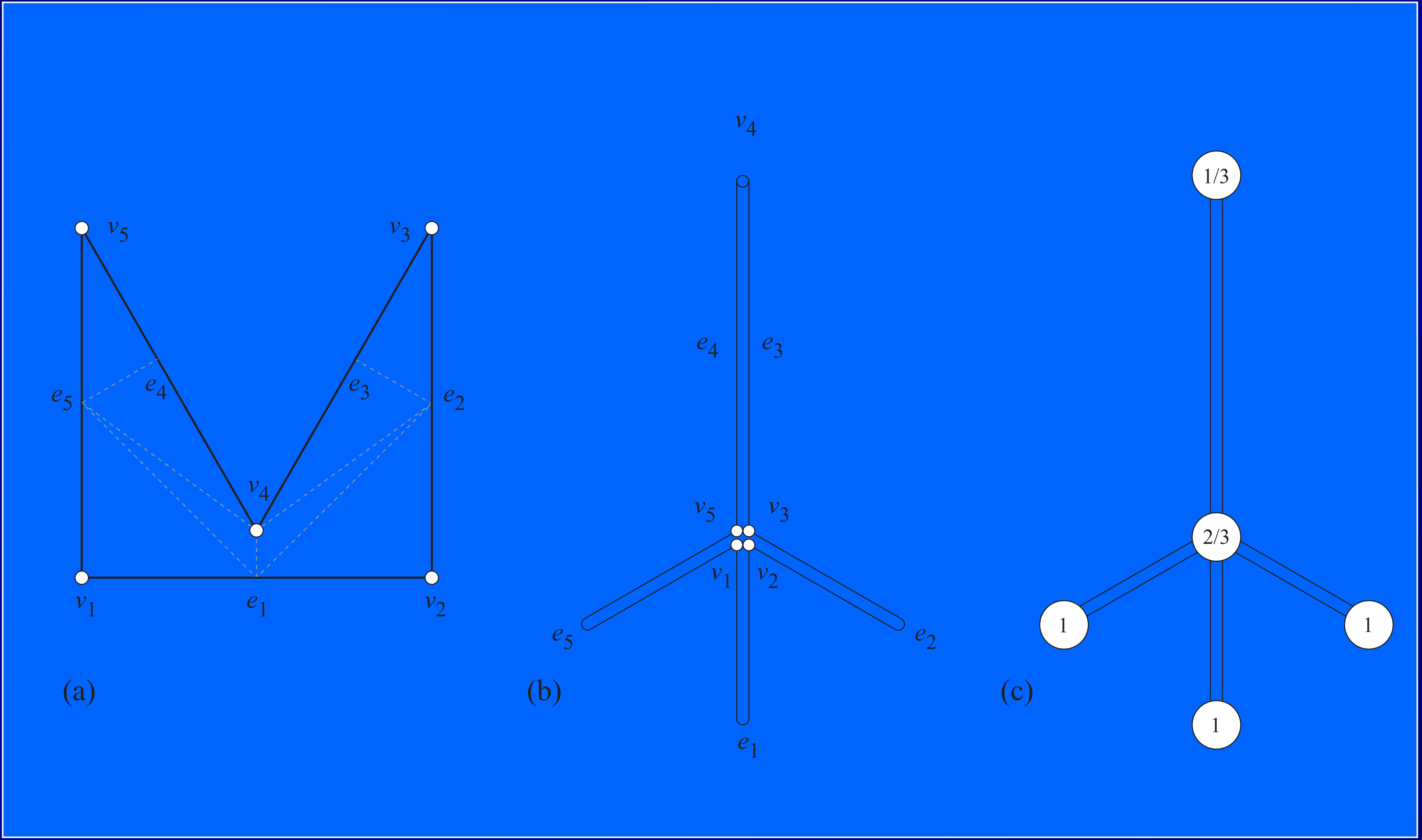


Video

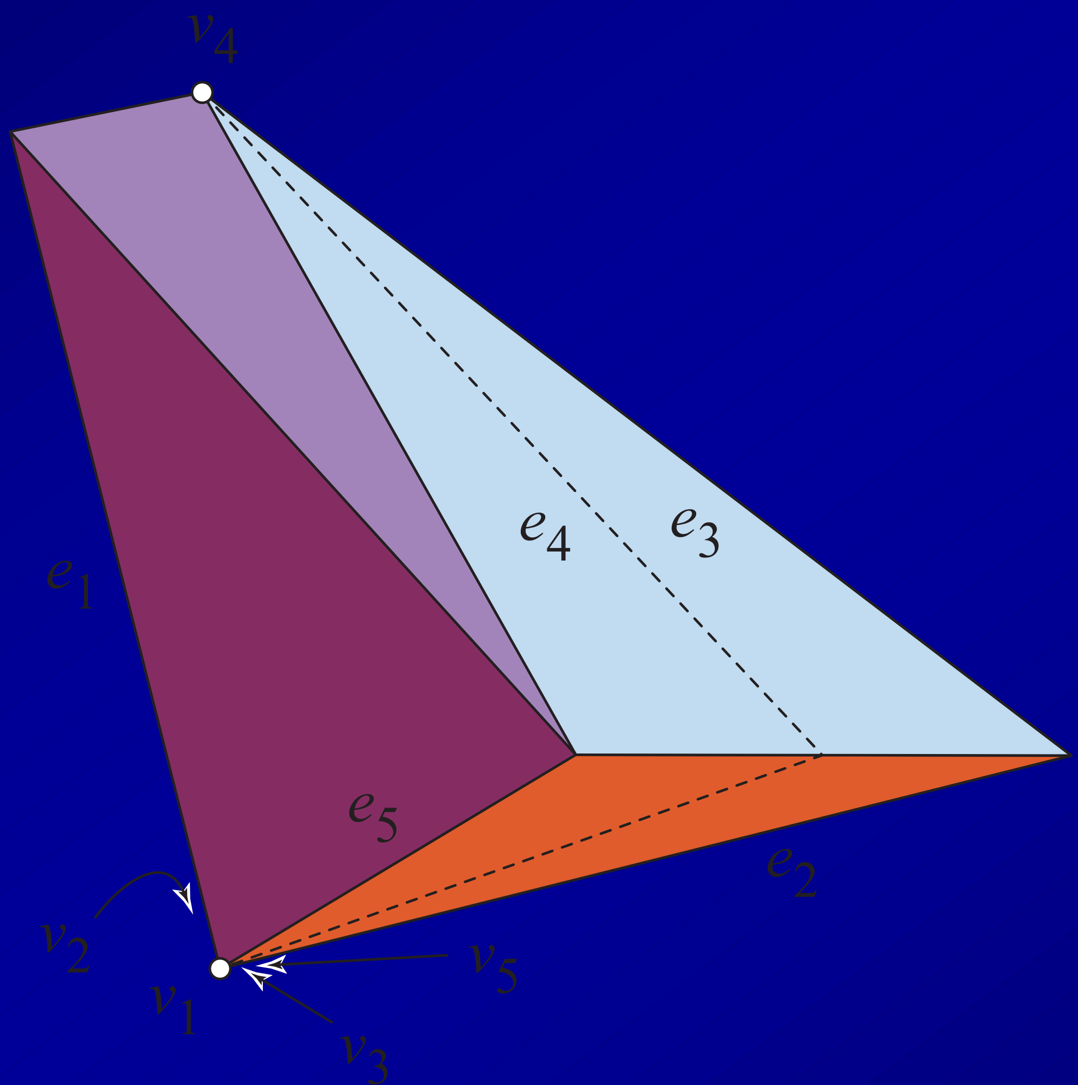


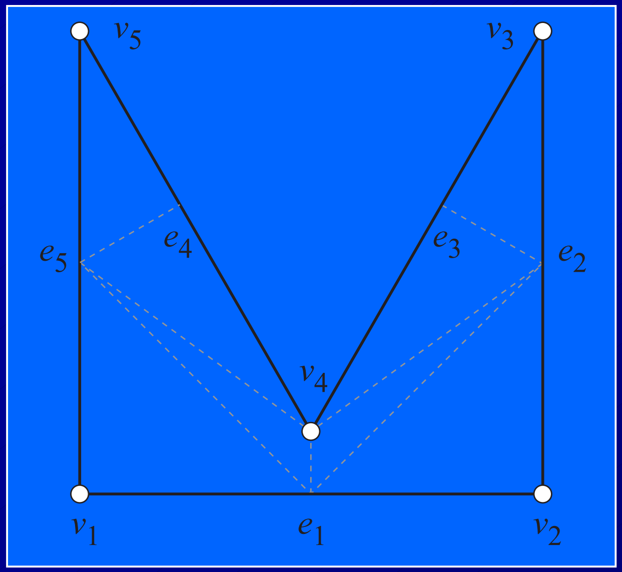
[Demaine, Demaine, Lubiw, JOR, Pashchenko (Symp. Computational Geometry, 1999)]

Gluing Trees



Folding of nonconvex pentagon

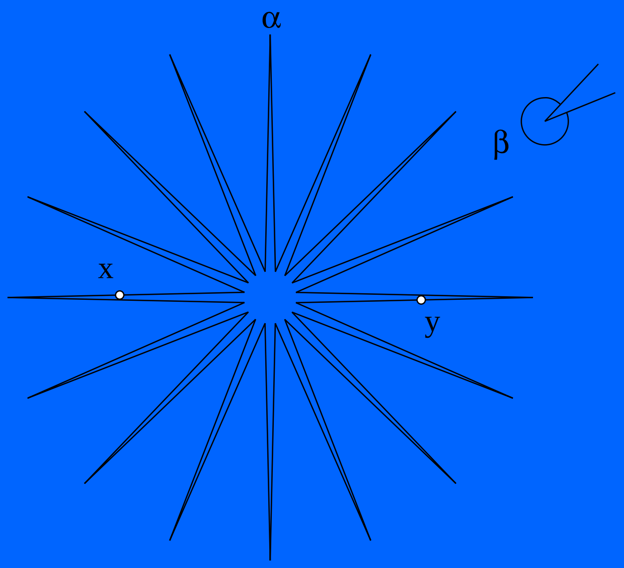




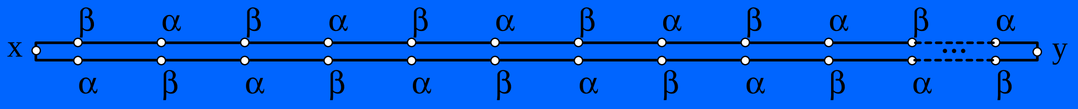
(a)

Exponential Number of Gluing Trees

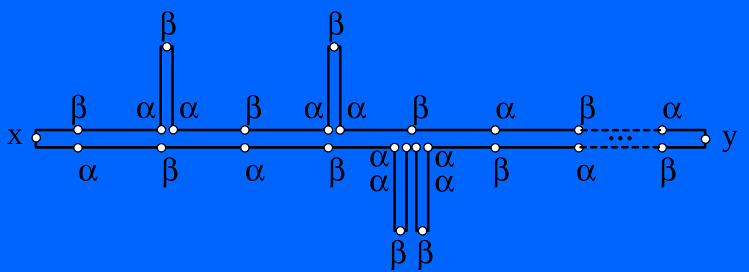
(a)



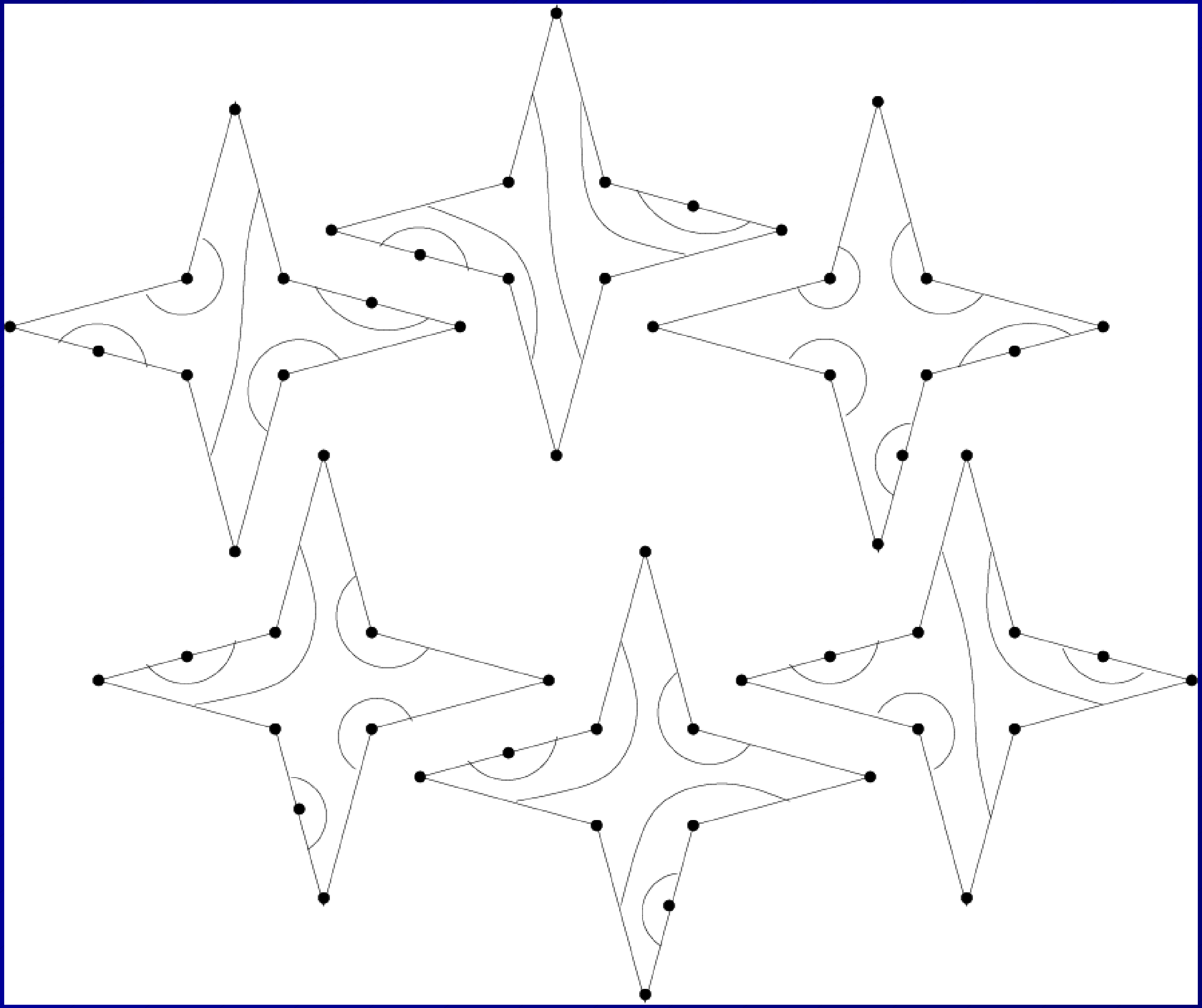
(b)



(c)



Exponential Number of Gluing Trees



General Gluing Algorithm

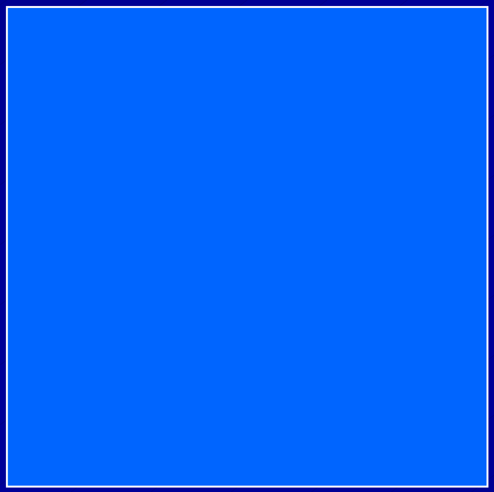
- No edge-to-edge assumption.
- Implementations: Anna Lubiw, Koichi Hirata (independently)
- Exponential-time, dynamic programming flavor.

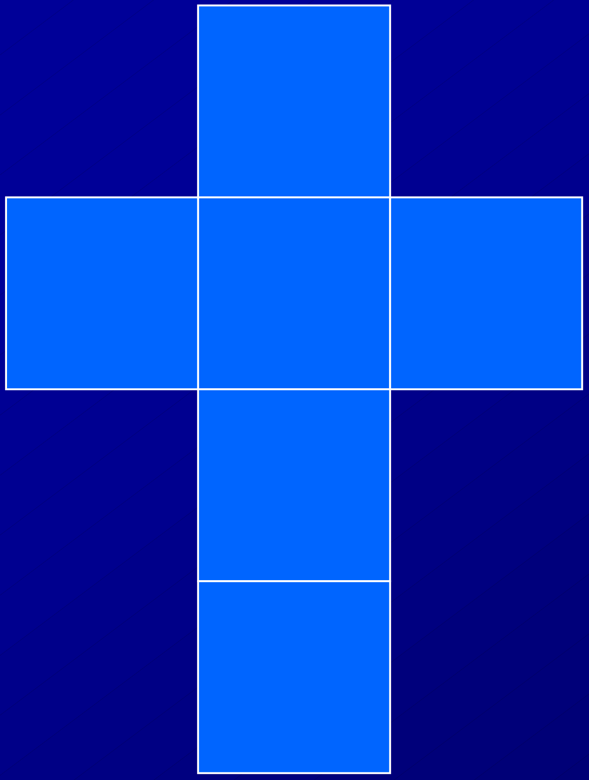
Open: Polynomial-time Folding Decision Algorithm

Given a polygon P of n vertices, determine in time polynomial in n if P has an Aleksandrov folding, and so can fold to some convex polyhedron.

Two Case Studies

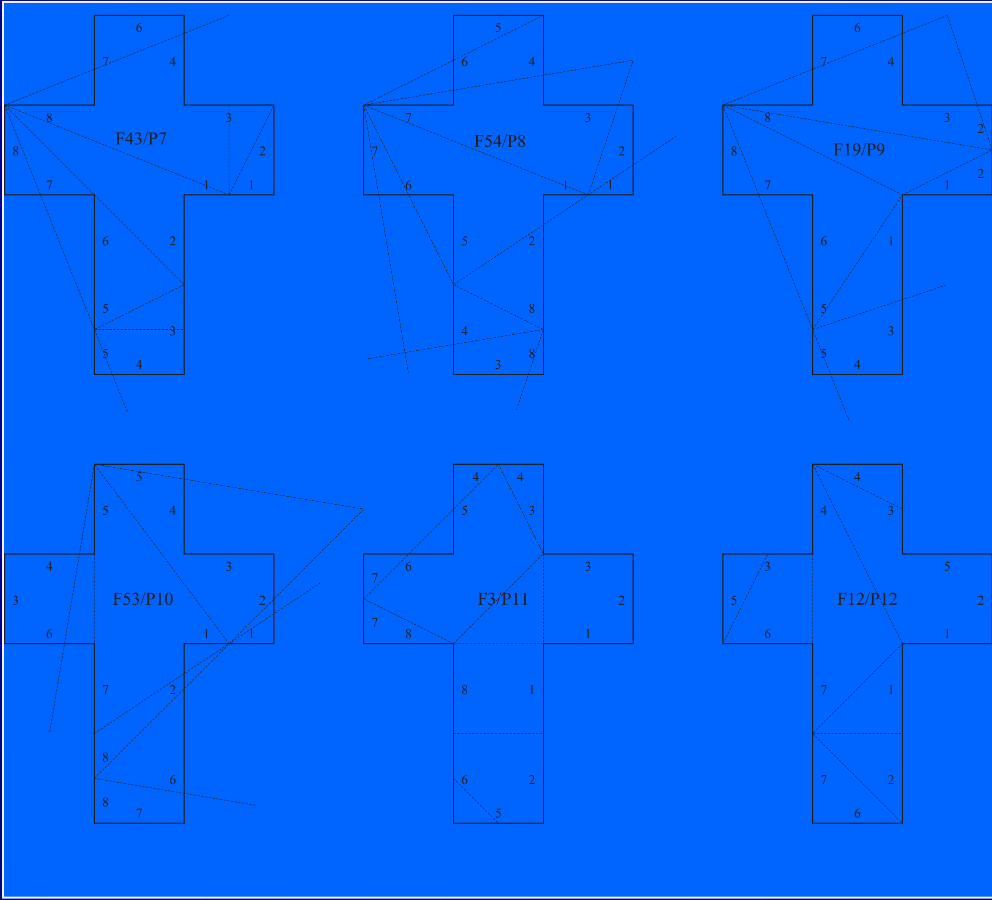
- The Latin Cross
- The Square



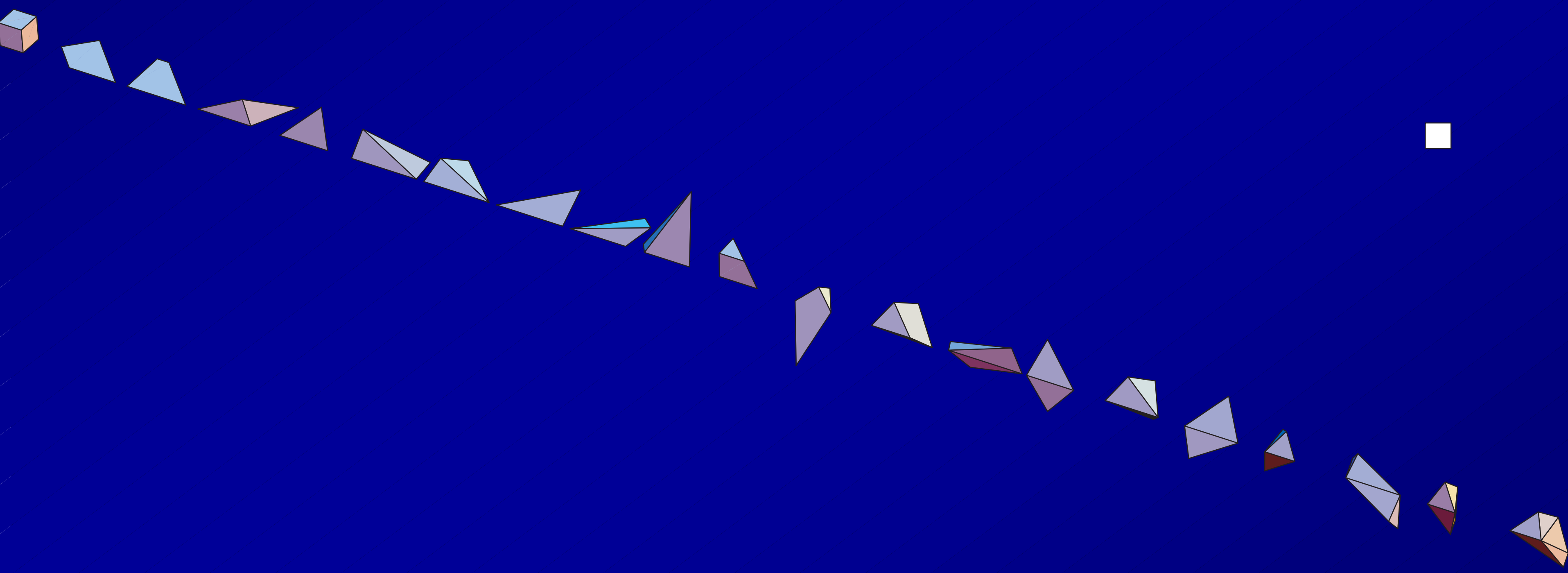


Folding the Latin Cross

- 85 distinct gluings
- Reconstruct shapes by ad hoc techniques
- 23 incongruent convex polyhedra

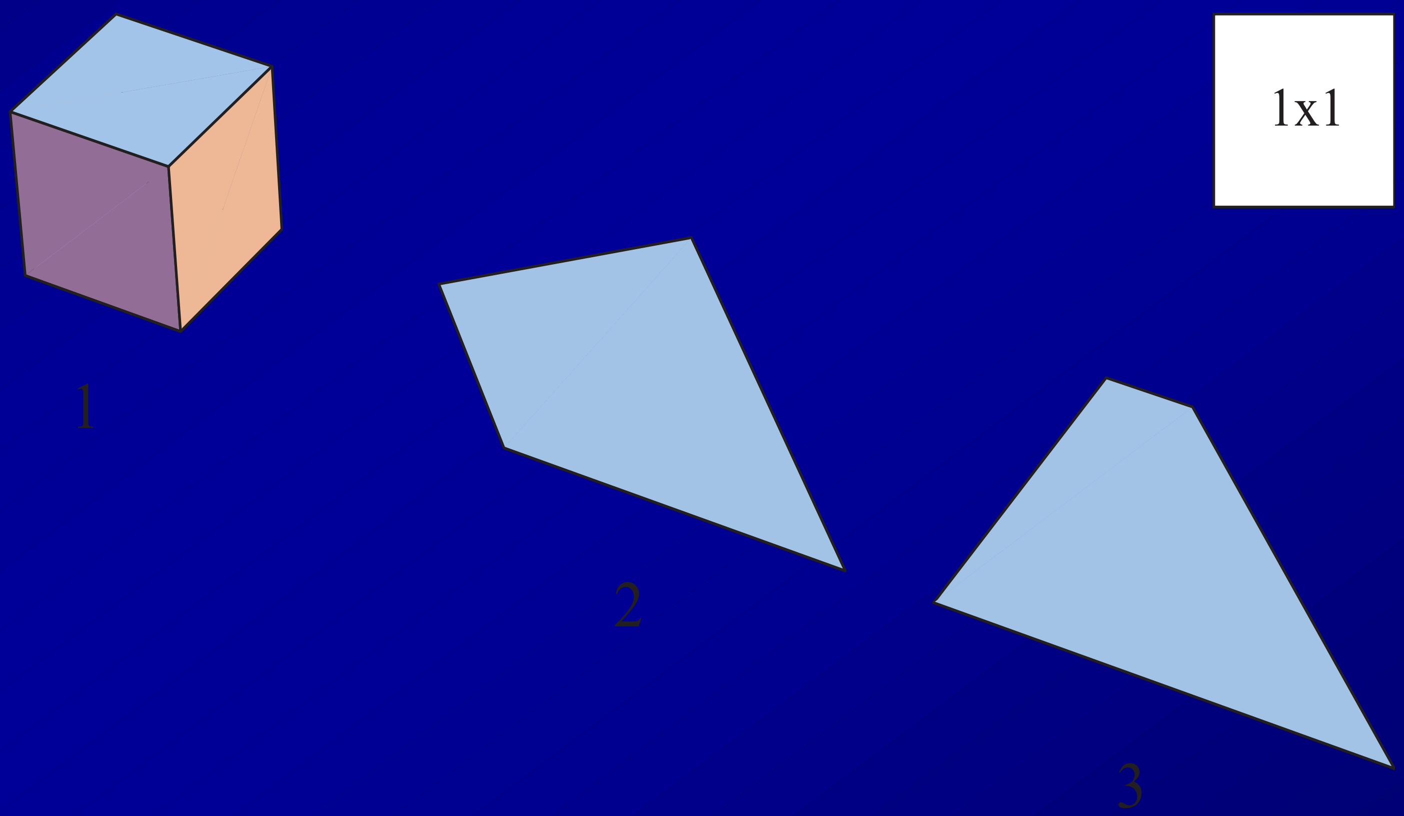


The 23 convex polyhedra foldable from the Latin cross

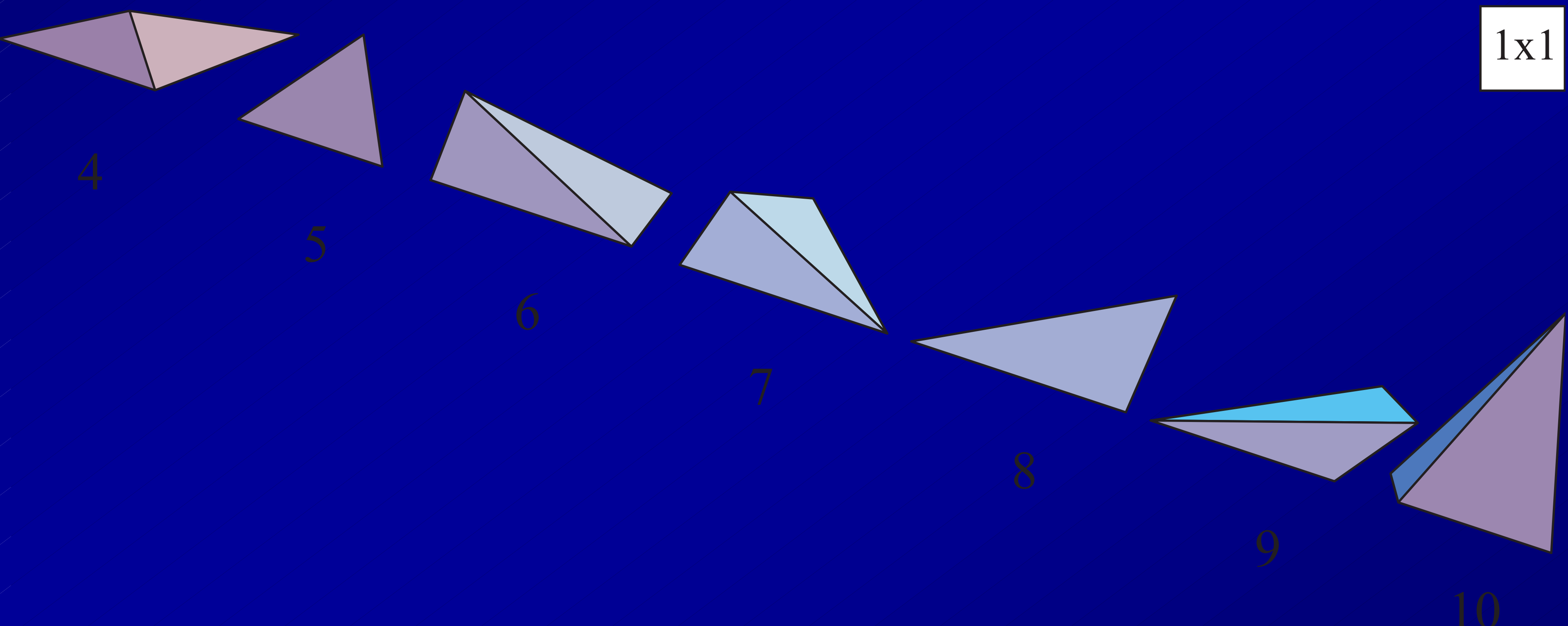


Sasha Berkoff, Caitlin Brady, Erik Demaine, Martin Demaine,
Koichi Hirata, Anna Lubiw, Sonya Nikolova, Joseph O'Rourke

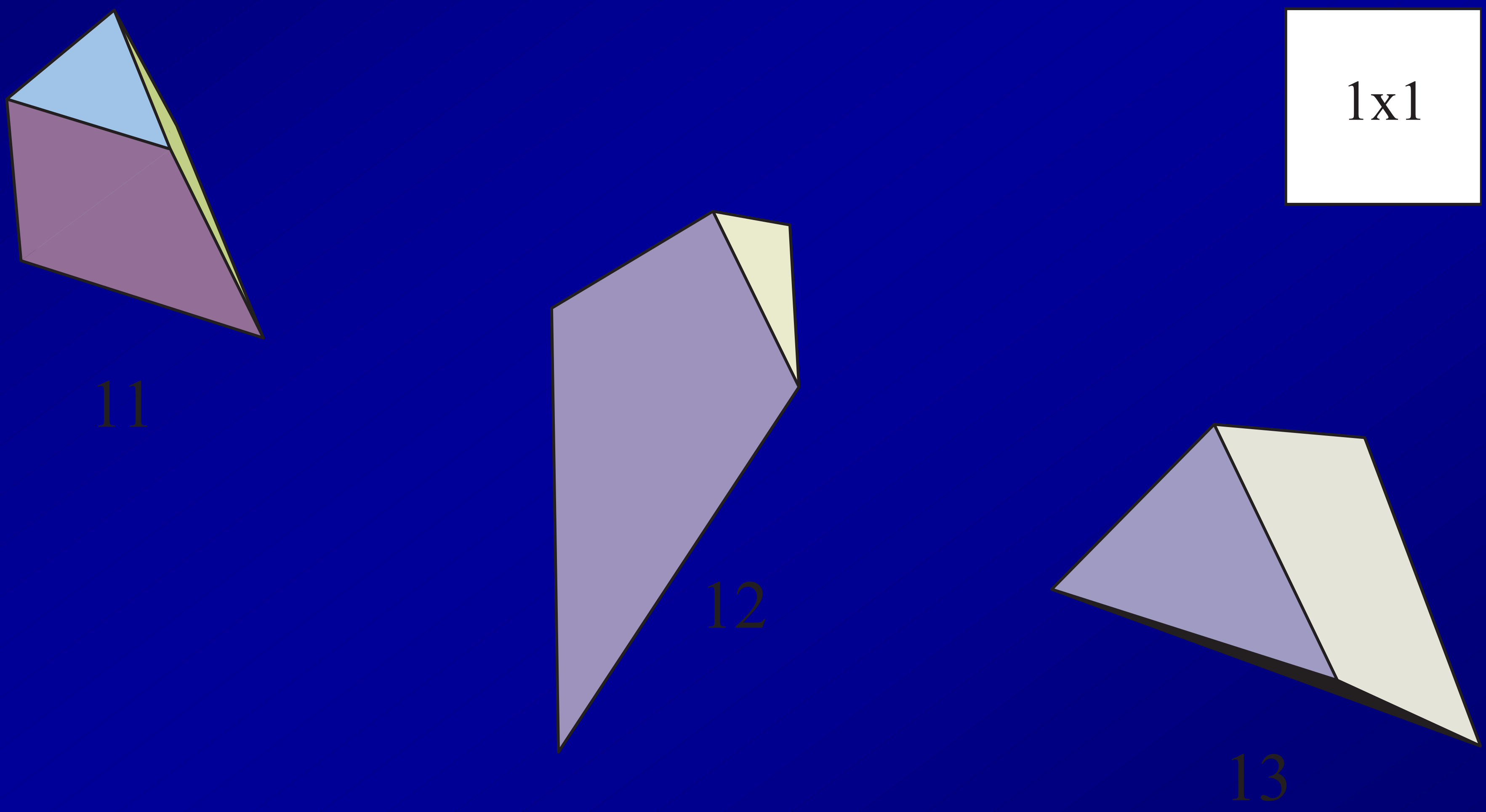
Cube + Flat Quadrilaterals



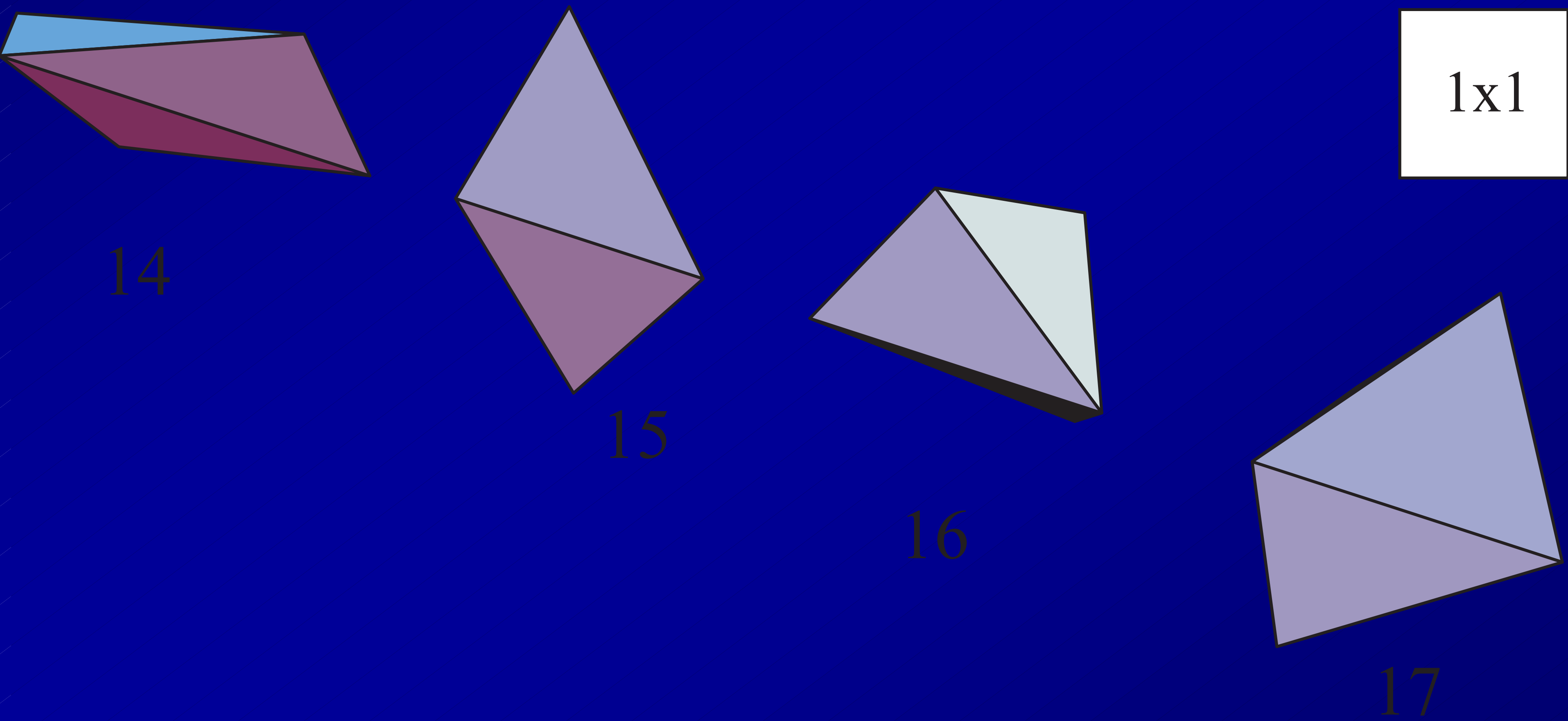
Latin cross Tetrahedra



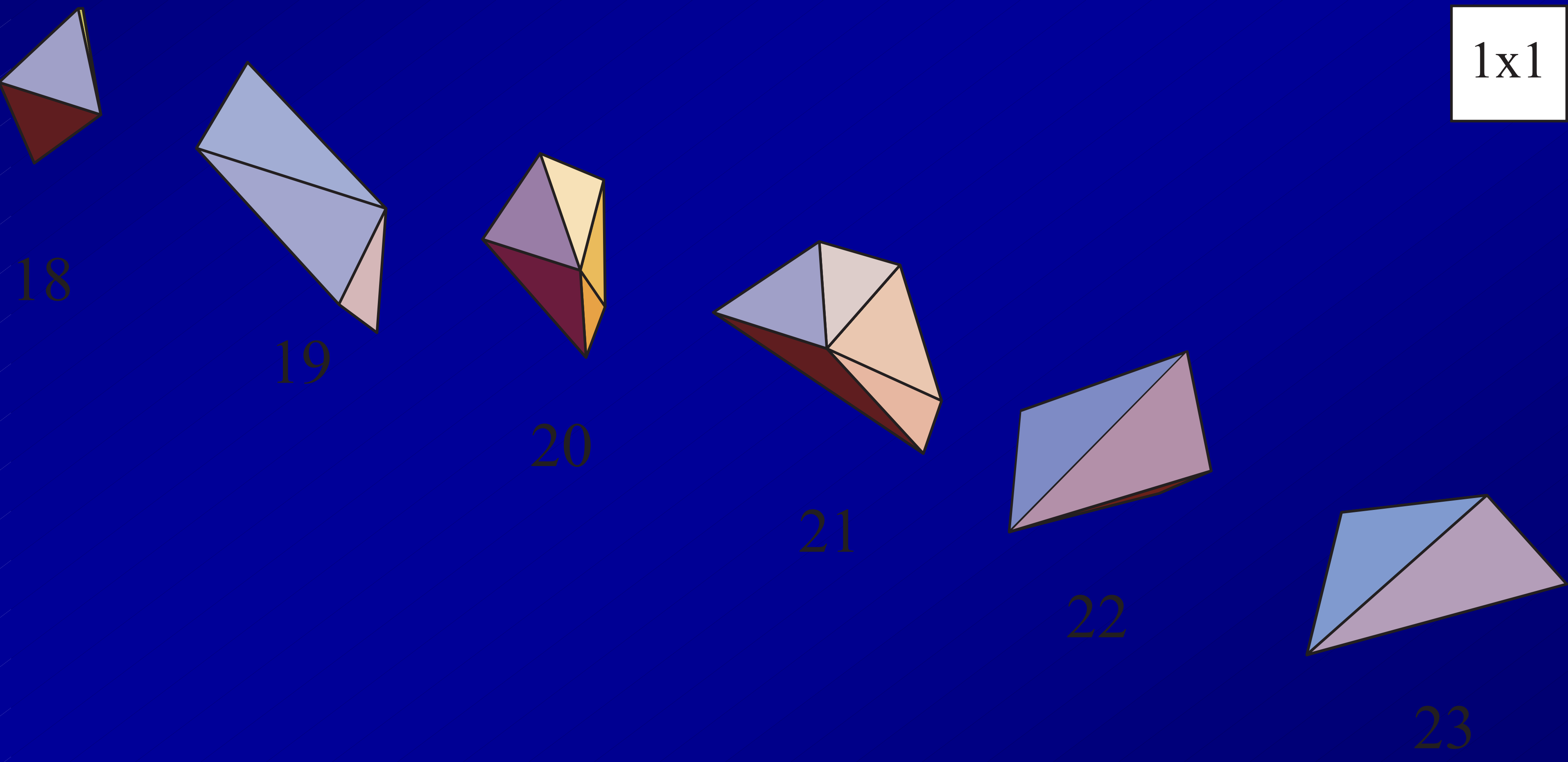
Latin cross Pentahedra



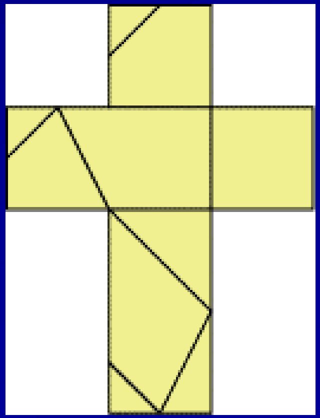
Latin cross Hexahedra

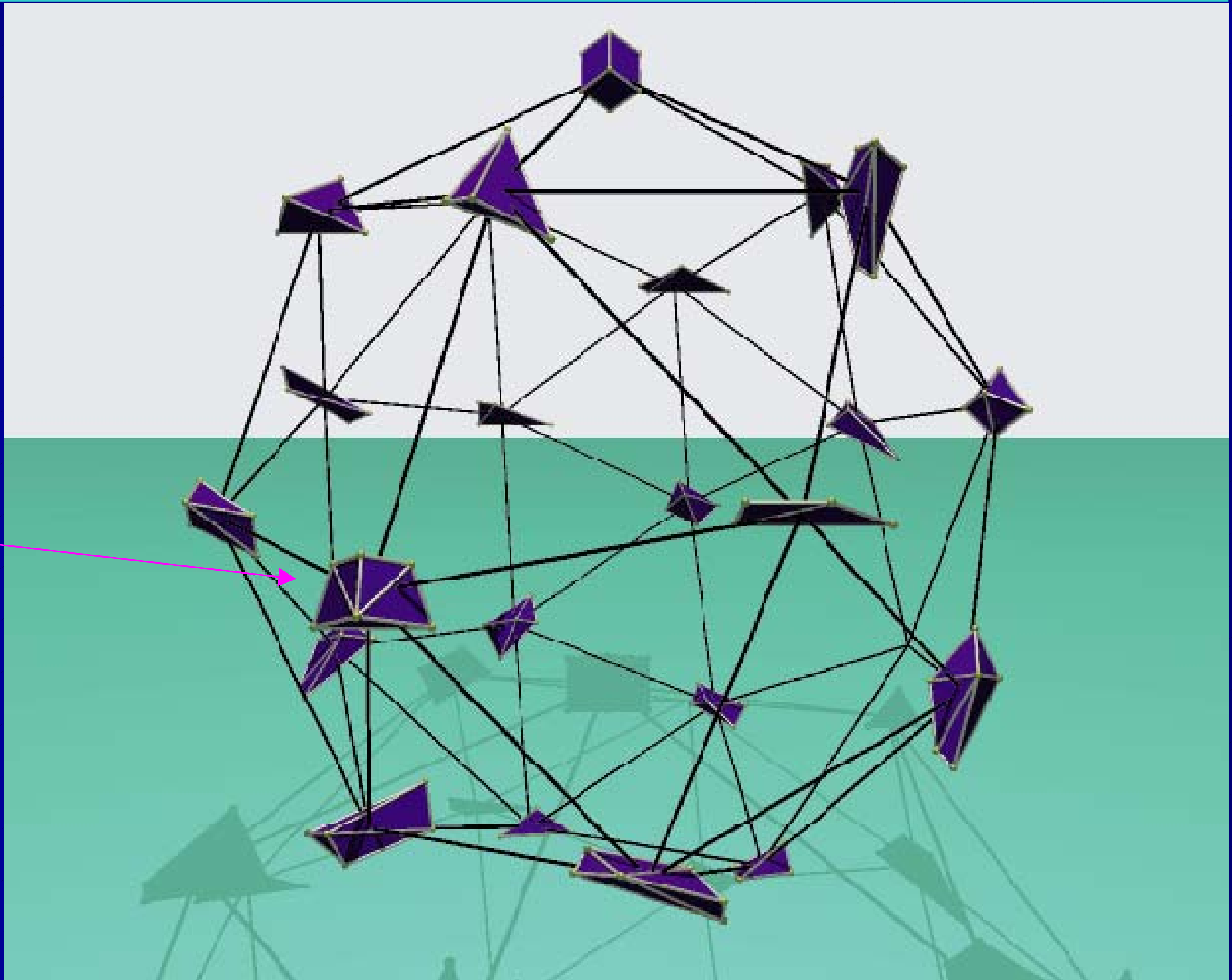


Latin cross Octahedra

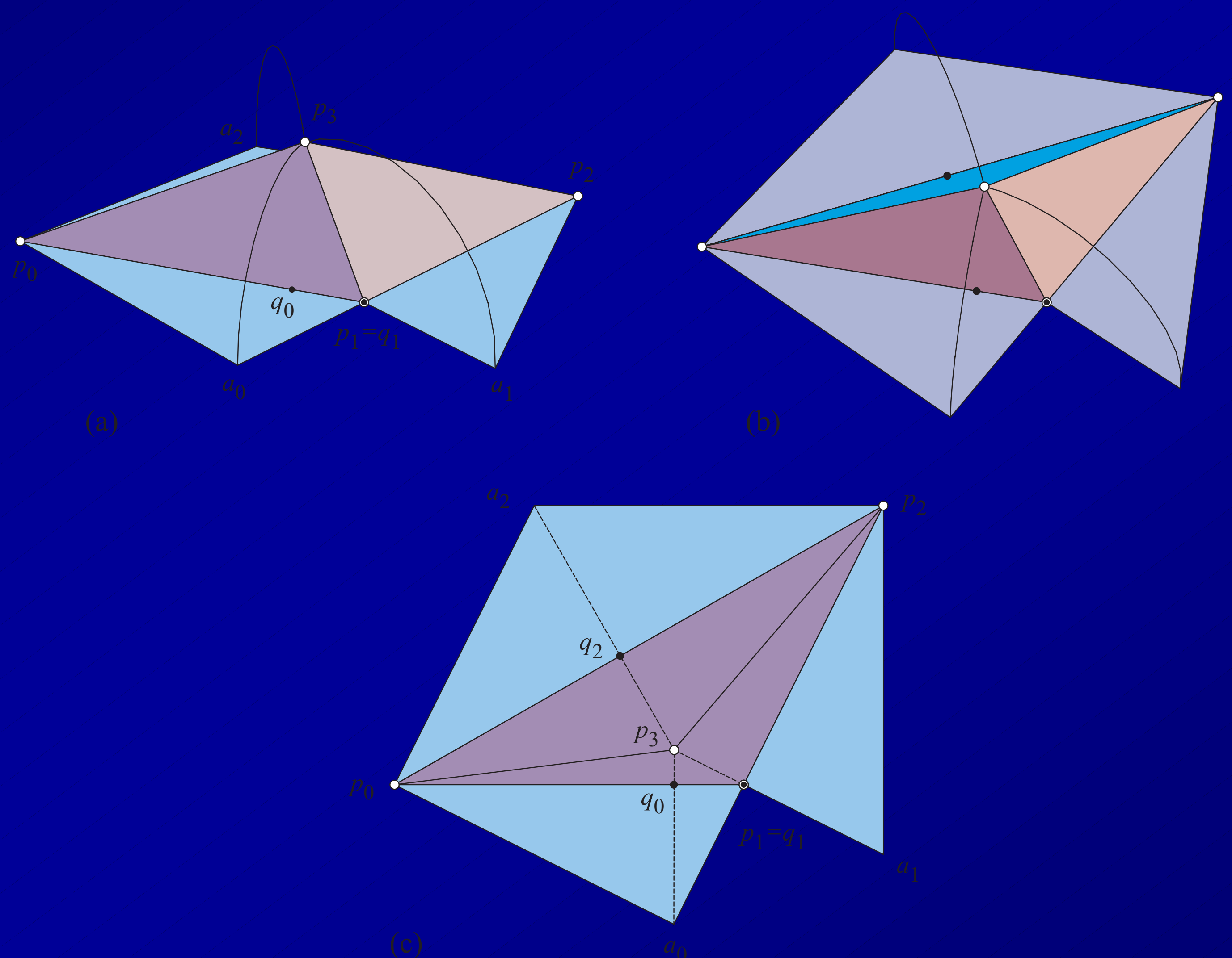


23 Latin Cross Polyhedra

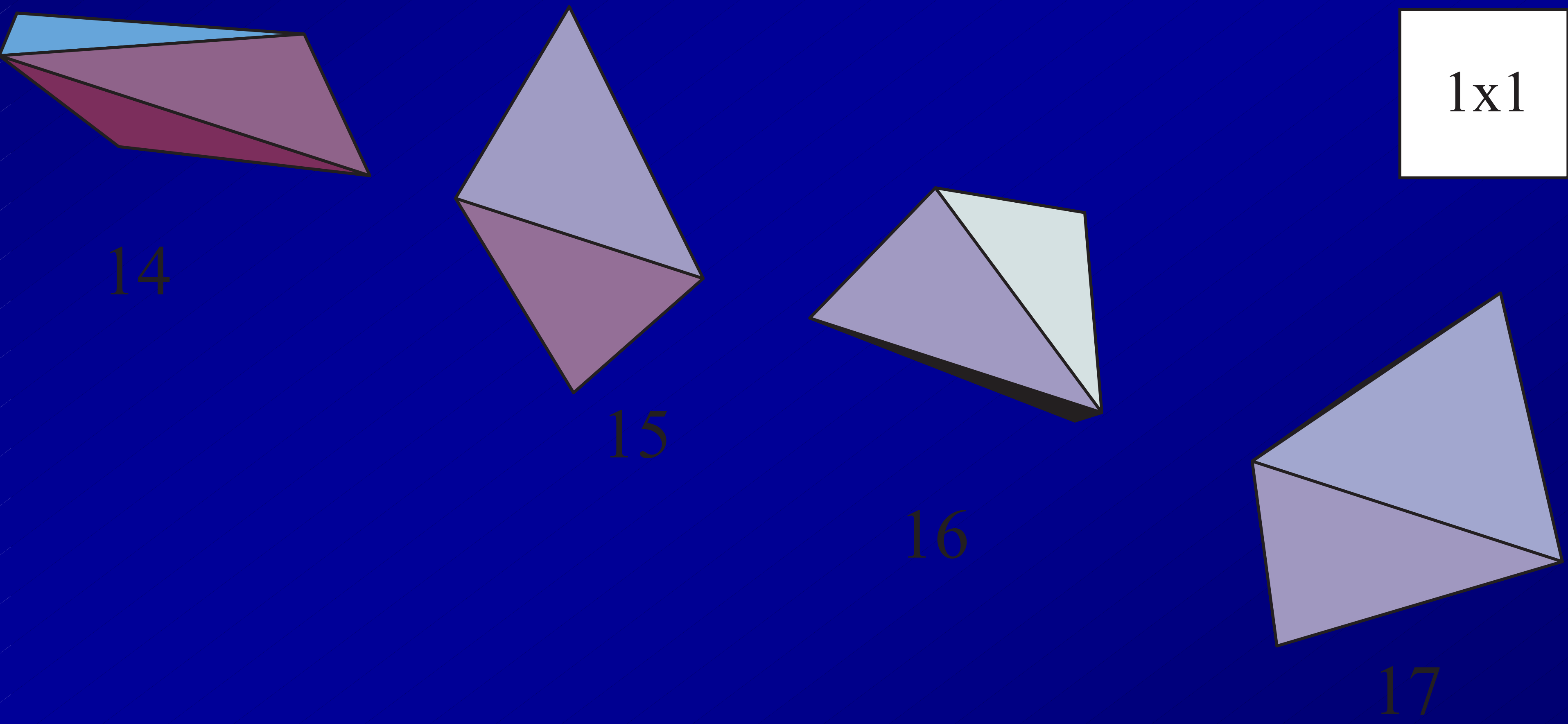




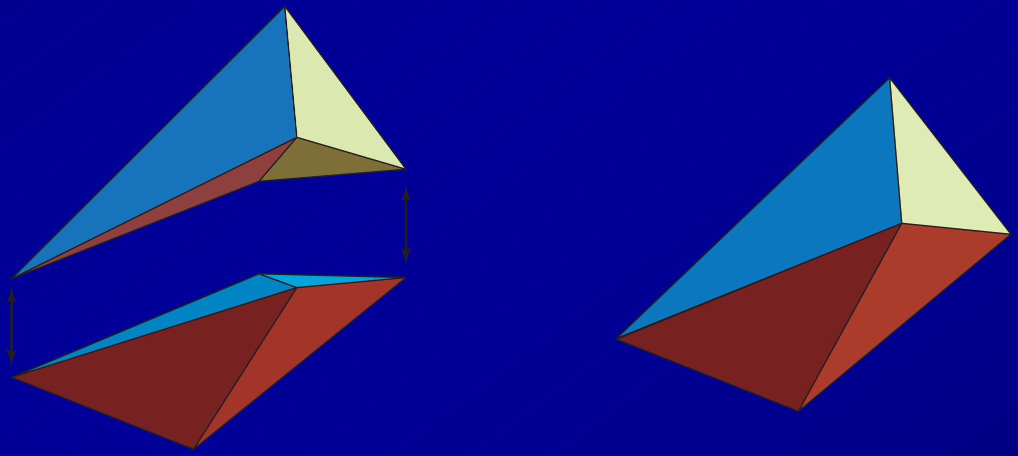
Reconstruction of Tetrahedron



Latin cross Hexahedra



Octahedron Reconstruction

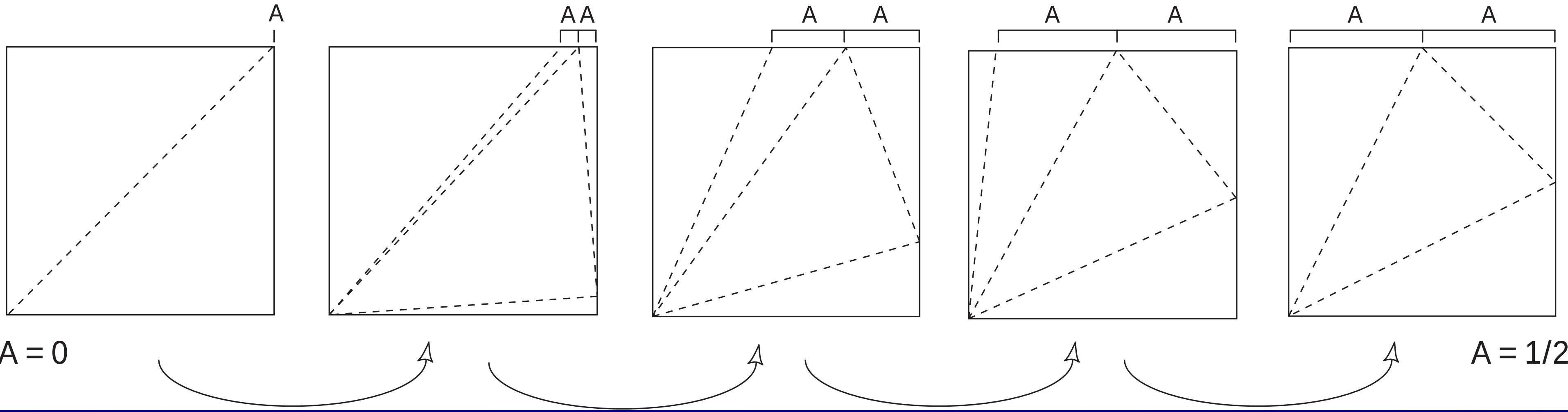


(a) (b)

Foldings of a Square

- Infinite continuum of polyhedra.
- Connected space

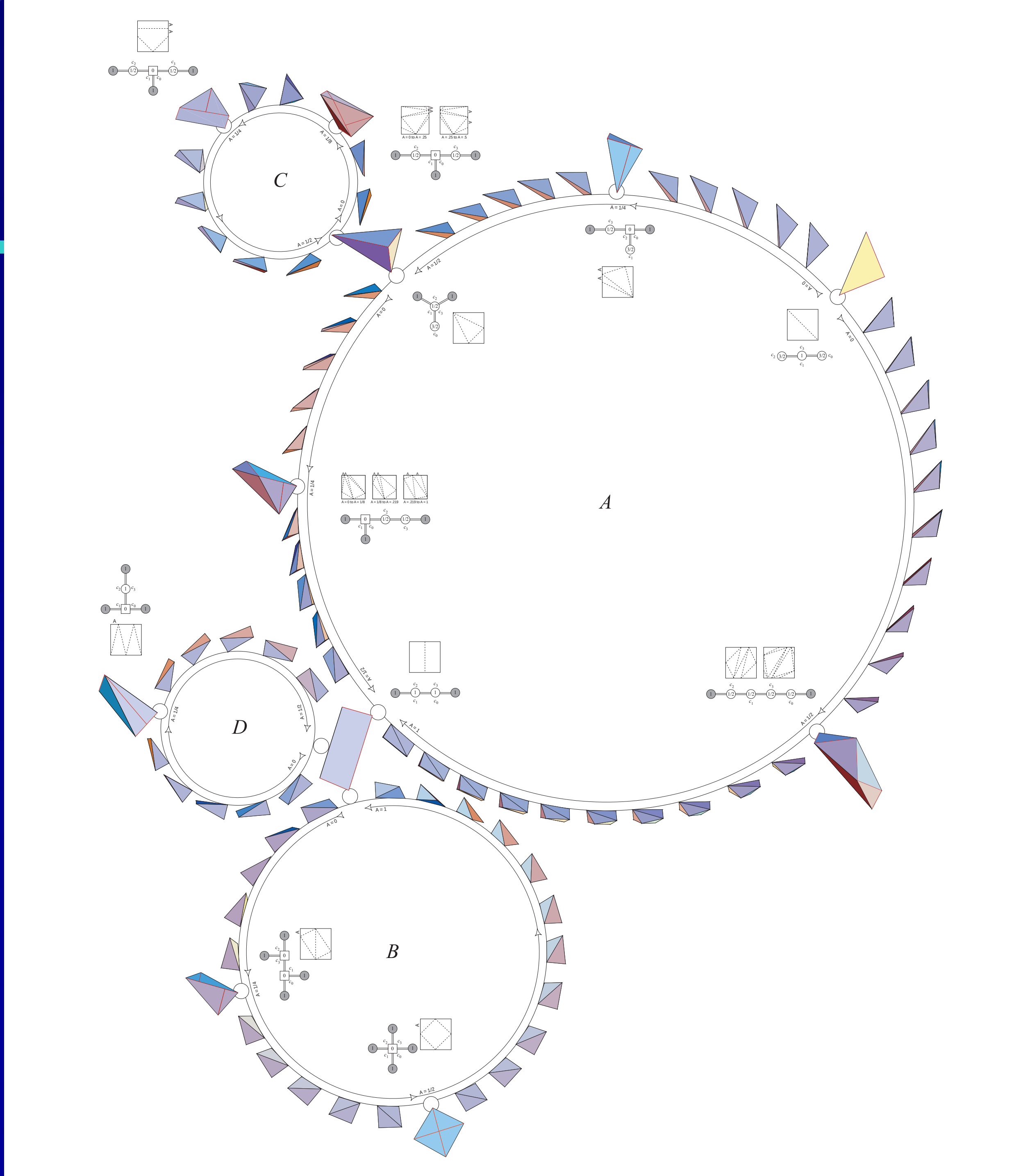
Creases



As *A* varies in [0,½],
the polyhedra vary between
a flat triangle and a symmetric tetrahedron.

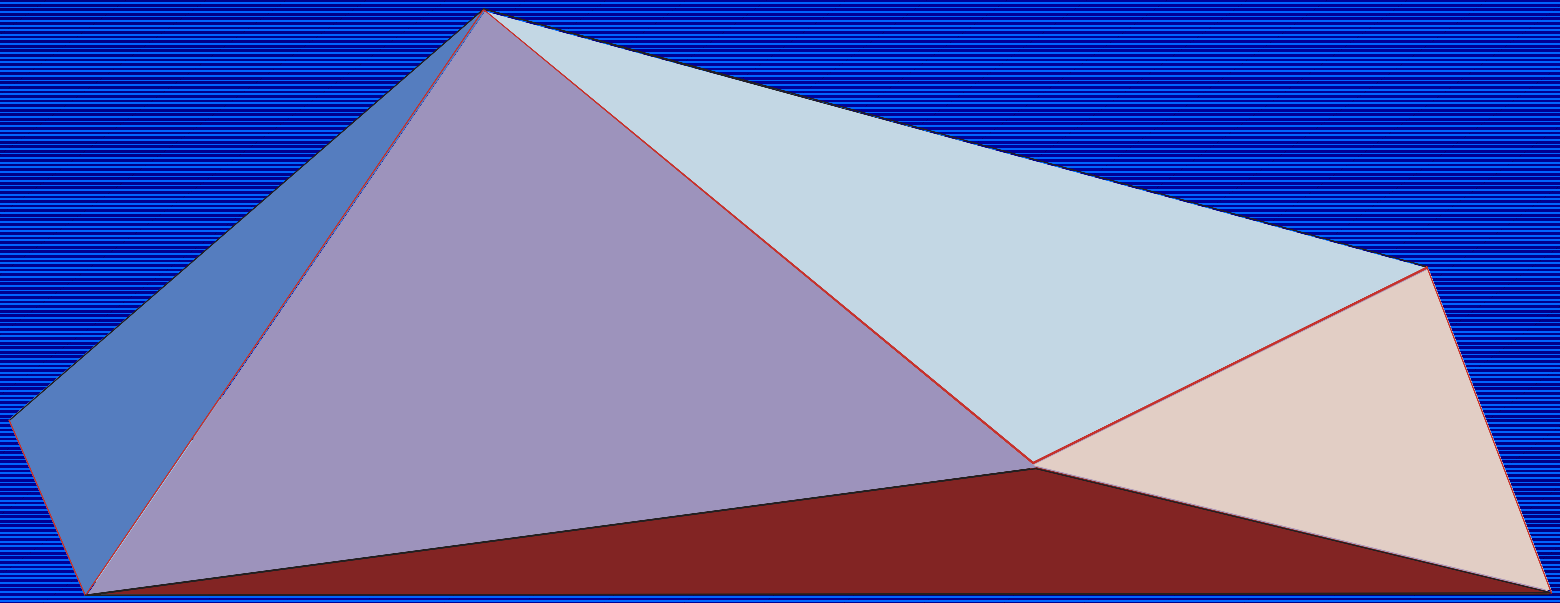
Nine Combinatorial Classes of Polyhedra foldable from a Square

- Five nondegenerate polyhedra:
  - Tetrahedra.
  - Pentahedra: 5 vertices and a single quadrilateral face,
  - Pentahedra : 6 vertices and three quadrilateral faces (and all other faces triangles).
  - Hexahedra: 5-vertex, 6-triangle polyhedra with vertex degrees (3,3,4,4,4).
  - Octahedra: 6-vertex, 8-triangle polyhedra with all vertices of degree 4.
- Four flat polyhedra:
  - A right triangle.
  - A square.
  - A $1 \times \frac{1}{2}$ rectangle.
  - A pentagon with a line of symmetry.



Dynamic Web page

Max Volume Polyhedron



Question due to Joseph Malkevitch , Feb 2002

Open: Fold/Refold Dissections

Can a cube be cut open and unfolded to a polygon that may be refolded to a regular tetrahedron?

[M. Demaine 98]