GEOMETRIC COMPLEXITY OF SPECIAL LAGRANGIAN T^2-CONES

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OUTLINE

- What is Special Lagrangian (SLG) geometry, and why study it?
- What are the motivating questions, and why are cones important for them?
- SLG T^2-cones via integrable systems methods.
- How to derive differential-geometric information from the integrable systems data.

If time permits I will also try to explain the following.

- How to make sense of "Some cones are more common that others." (generic singularities in almost Calabi-Yaus).

- Answers to "What are the most common cones?" (cones with smaller Legendrian index).

Calibrations – Definitions

A *calibrated geometry* is a distinguished class of minimal submanifolds associated with a differential form.

- A *calibrated form* is a closed differential p-form ϕ on a Riemannian manifold (M, g) satisfying

$$\phi \leq vol_g \qquad (*)$$

- For $m \in M$ associate with ϕ the subset $G_m(\phi)$ of oriented p-planes for which equality holds in $(*)$ – the *calibrated* planes.

- A submanifold *calibrated* by ϕ is an oriented p-dim submanifold whose tangent plane at each point m lies in the subset $G_m(\phi)$ of distinguished p-planes.

Lemma: (Harvey–Lawson) Calibrated submanifolds minimize volume in their homology class.

Special Lagrangian Calibration on $\mathbb{C}^n$

On $\mathbb{C}^n$ with standard complex coordinates, let

$$\alpha = \mathrm{Re}(\Omega), \text{ where } \Omega = dz^1 \wedge \ldots \wedge dz^n.$$

- α is a calibrated form.
- An α-calibrated submanifold is Lagrangian w.r.t. standard Kähler form $\omega = \sum dx^i \wedge dy^i$.
- An α-calibrated plane may be obtained from standard "real" $\mathbb{R}^n \subseteq \mathbb{C}^n$ by action of $A \in \mathrm{SU}(n)$. Thus the name *special* Lagrangian calibration.
- An alternative characterization of an SLG plane is:

  A real n-plane P in $\mathbb{C}^n$ is SLG for some choice of orientation iff ω and $\beta = \mathrm{Im}(\Omega)$ both restrict to zero on P.

Examples of Special Lagrangian submanifolds in $\mathbb{C}^n$

SLG level sets (Harvey – Lawson 1982)
(explicit examples with symmetries)

$F : \mathbb{C}^3 \to \mathbb{R}^3$
$F = \left(|z_1|^2 - |z_2|^2,\ |z_1|^2 - |z_3|^2,\ \mathrm{Im}(z_1 z_2 z_3)\right)$

$F^{-1}(c)$ is a (possibly singular) SLG 3-fold invariant under a T^2 action.

- For a generic c, $F^{-1}(c)$ is nonsingular and diffeomorphic to $T^2 \times \mathbb{R}$.
- For 1-dimensional set of critical values of c, $F^{-1}(c)$ is diffeomorphic to $S^1 \times \mathbb{R}^2$.
- $F^{-1}(\mathbf{0})$ is a singular cone.

SLG geometry in Calabi-Yau and almost Calabi-Yaus

Need manifolds with an analogue of the constant coeff $(n, 0)$-form $\Omega = dz^1 \wedge \ldots \wedge dz^n$.

Constant coeff $\rightarrow$ parallel for ∇, Levi-Civita connection.

Existence of nonzero parallel $(n, 0)$-form on M implies M is a Kähler manifold with vanishing Ricci tensor. *i.e.*, a Calabi-Yau manifold. Then $\mathrm{Re}(\Omega)$ is a calibrated form.

An almost Calabi-Yau manifold is a Kähler manifold with a nowhere zero holomorphic $(n, 0)$-form Ω which is not necessarily a *parallel* form. Implies metric not in general Ricci flat.

Def^n: L is *Special Lagrangian* if ω and $\mathrm{Im}(\Omega)$ both restrict to zero and $\mathrm{Re}(\Omega) > 0$ on L.

SLG geometry in almost Calabi-Yaus

L is usually neither *calibrated* nor *minimal* w.r.t. the chosen Kähler metric g !

L is calibrated for a conformally equivalent metric $\tilde{g}$.

Many nice features of SLG on CY manifolds, still hold on almost CY manifolds. *e.g.* lots of local solutions, unobstructed deformation theory for compact smooth SLG submanifolds.

Deformation theory (McLean 94,Joyce 00): L compact nonsingular SLG submanifold in CY or almost CY manifold. Then deformation theory is always unobstructed and dimension is purely topological, equal to $b^1(L)$.

Nicer answer than in complex geometry!

Basic reason: for L Lagrangian NL and TL are isomorphic. Allows us to convert extrinsic into instrinsic data. Infinitesimal SLG deformations $\longleftrightarrow$ harmonic 1-forms on L.

Advantages of SLG geometry in almost CY manifolds

- Calabi-Yau mfds come in finite dimensional families (dim $= h^{1,1} + h^{2,1} + 1$ in 3d). Almost Calabi-Yau manifolds come in *infinite* dimensional families.
- Choice of a *generic* almost Calabi-Yau structure is a much more powerful assumption than generic Calabi-Yau structure.
- Much easier to construct explicit examples of almost Calabi-Yau manifolds. Compact Calabi-Yau manifolds are known by analytic existence results, not explicit construction.

Question: Does studying SLG submanifolds tell us anything about the ambient manifold?

A global problem in SLG geometry

Counting SLG homology spheres:

A rational homology sphere S has $b_1(S) = 0$. Deformation theory implies that an SLG homology sphere is infinitesimally rigid.

Can we count the # of SLG homology spheres in a given homology class?

SLG analogue of Gromov-Witten invariants – which count the number of J-holo curves in a given homology class.

Main issues:

A. Compactness: Are SLG homology spheres isolated?

B. Deformation invariance: Does count give a number which is invariant of underlying CY structure? (Is it symplectic invariant?)
Need to understand behaviour in 1-parameter families of CYs, in which S can become singular.

Global problems in SLG geometry II

2. Strominger-Yau-Zaslow (1996) conjecture:

Physical reasoning strongly suggested SLG torus fibrations could provide differential-geometric explanation of mirror symmetry.

Given a pair of mirror Calabi-Yau manifolds M and $\widehat{M}$, M should have a (singular) SLG torus fibration $f : M \to B$ which should determine $\widehat{M}$ by "dualizing" the fibration f.

Main issues:

A. Probably should only hold in some limiting regimes of CY moduli space.

B. Still no rigorous mathematical version of conjecture.

C. Understanding how to deal with the singular fibres.

SLG Cones and Their Links

Def^n: A *cone* $C \subset \mathbb{C}^N$ is a subset invariant under all dilations.

For $\Sigma \subset S^{2N-1} \subset \mathbb{C}^N$, let $C(\Sigma)$ denote the cone on Σ:

$$C(\Sigma) = \{tx : t \geq 0, x \in \Sigma\}$$

Def^n: A cone is *regular* if $C = C(\Sigma)$ where Σ is compact, connected, embedded and oriented submanifold of S^{2N-1}.

Call Σ the *link* of the regular cone C.

Prop^n: C is SLG in $\mathbb{C}^N$ (up to a phase) $\Longleftrightarrow$ Σ is minimal and Legendrian in S^{2N-1}.

Legendrian is the odd-dimensional analogue of Lagrangian.

Known Singularities I

$\underline{n=2}$: just have unions of 2-planes (link is a 1d minimal submanifold of S^3 *i.e.*, a union of geodesics $\Rightarrow$ corresponding cones are planes)

$\underline{n=3}$: (Harvey & Lawson, 1982)
$(z_1, z_2, z_3) \in \mathbb{C}^3$ s.t.
$|z_1|^2 = |z_2|^2$
$|z_1|^2 = |z_3|^2$
and $\text{Im}(z_1 z_2 z_3) = 0$.

Cone over union of 2 Clifford tori, $\Sigma_\pm$.

$$C = C_+ \cup C_- = C(\Sigma_+) \cup C(\Sigma_-)$$

$\Sigma_\pm$ unique <u>flat</u> minimal Legendrian torus in S^5 up to unitary equivalence. (invariant under $T^2 \subset \text{SU}(3)$).

1982 – late 90s, no other examples known.

Known Singularities II

2000 (Haskins): $\exists$ countably infinite families of distinct SLG cones with:

1. $\Sigma \simeq T^2$.

2. Σ is invariant under some $S^1 \subseteq \text{SU}(3)$.

Have explicit descriptions in terms of elliptic functions and elliptic integrals. Can give very detailed description of the geometry using these explicit descriptions (e.g. curvature, conformal structures, area).

These include ‘almost flat’ examples and examples of arbitrarily large area $A(\Sigma)$.

SLG cones and Integrable Systems I

2002 (McIntosh): Minimal Lagrangian tori in $\mathbb{C}P^2$ (up to congruence) are in one-to-one correspondence with certain algebro-geometric "spectral data", $(X, \lambda, \mathcal{L}, \mu)$.

- X is a compact Riemann surface of genus $g = 2p$, and $\mathcal{L}$ is a holo line bundle on X.
- Condition that spectral data gives rise to torus puts strong restrictions on the spectral curve – a kind of "rationality condition".
- $p = 0 \Rightarrow$ must get Clifford torus.
  $p = 1 \Rightarrow$ get S^1-invariant tori (Haskins).
  $p = 2 \Rightarrow$ have examples of Joyce.
- Not obvious that for $p > 2$ there are any spectral curves giving rise to tori.

SLG cones and Integrable Systems II

Aug 2003 McIntosh-Carberry proved the following (announced in Nov 02). Go to Emma's talk on Wednesday for details.

For each $p > 2$, there exist countably many spectral curves of genus $2p$, each of which gives rise to continuous families of minimal Lagrangian tori.

- Implies there is a whole zoo of SLG T^2-cones in $\mathbb{C}^3$. Unrealistic to expect a complete classification.
- Very difficult to derive differential geometric information, *e.g.* area, from spectral curve information.

Differential geometric information from integrable systems data

Similar spectral curve constructions exist for harmonic tori into compact Lie groups and other rank 1 symmetric spaces. Other geometrically interesting examples include:

- constant mean curvature tori in $\mathbb{R}^3$ (Gauss map is non-conformal harmonic map to S^2)
- minimal tori in S^3
- Willmore tori in S^3 and S^4 (work in progress of TU-Berlin school)

Evidence from computer experiments with CMC tori that:

large spectral curve genus

$\Rightarrow$ geometrically complicated torus

$\Rightarrow$ e.g. large area or energy, v. unstable.

Differential geometric information from integrable systems data II

Suggests spectral curve genus g gives a lower bound for the geometric complexity of a SLG T^2-cone (S^1-invariant examples show no upper bound for geometric complexity is possible in terms of g).

For instance, would like to prove

$$\text{Area}(T^2) \geq c(g),$$

for some positive increasing function c of the spectral curve genus g.

Ferus, Leschke, Pedit, Pinkall (2001) proved such a formula holds for minimal tori in S^3 using Quaternionic Holomorphic methods (long paper almost 90 pages).

- Would like to prove analogous results in minimal Lagrangian case.
- BUT, no obvious way to use quaternionic holomorphic methods.

Differential geometric information from integrable systems data III

Nevertheless, we can prove a SLG analogue of the results of FLPP.

Thm (Haskins 2003) Let C be a regular SLG T^2-cone in $\mathbb{C}^3$ with link Σ. Let $g = 2d$ denote the spectral curve genus of the associated minimal Lagn torus in $\mathbb{C}P^2$. Then for any $d \geq 3$

$$\text{Area}(\Sigma) > \frac{1}{3} d\pi.$$

Instead of quat holo methods we use methods from spectral geometry of Δ on compact surfaces.

Outline of proof of Thm

- Relate second variation of area of link Σ of cone C to Δ_Σ
- Relate spectral curve genus to properties of second variation operator
- Combine previous parts to obtain information about Spec(Δ_Σ).
- Use information about Spec(Δ_Σ) to get lower bound for Area(Σ).

Second Variation of Area on Σ and Δ_Σ – Legendrian Variations

Legendrian Neighborhood Th$^{\mathrm{m}}$: A neighbourhood of a Legendrian submanifold L of a contact manifold M is contact diffeomorphic to a neighbourhood of the zero section in $J^1(L)$ – the first Jet space of L – with its canonical contact structure.

Moreover, a section of $J^1(L)$ is Legendrian iff it is the 1-Jet associated to a function on L.

$$J^1(L) \sim \mathbb{R} \times T^*L$$

The contact form on J^1L is

$$dt - \pi^*\alpha,$$

where t is the coordinate on $\mathbb{R}$ and α Liouville 1-form on T^*L.

Legendrian section of $J^1(L)$ $\rightsquigarrow (f(x), x, df(x))$ some $f : L \to \mathbb{R}$

Note: function is global unlike Lagrangian case.

Second variation of a minimal Legendrian submanifold

Second variation operator of a minimal Legendrian submanifold $\Sigma \subset S^{2N-1}$ restricted to Legendrian variations.

Using the Legendrian Neighbourhood Th$^{\mathrm{m}}$, we can express it as operator on function on Σ. Nice answer . . .

$$\mathcal{J}_\Sigma = \Delta_\Sigma - 2N$$

(Δ w.r.t. metric induced on Σ by embedding in sphere)

Infinitesimal Deformations as a minimal Legendrian submanifold $\Longleftrightarrow f$ s.t. $\Delta f = 2Nf$.

$\Delta f = \lambda f$, $\lambda < 2N \Rightarrow$ Volume Decreasing infinitesimal deformations as Legn submanifold.

Legendrian index of a SLG cone $C = C(\Sigma)$

l-ind(C) := # of e-values, counting multiplicity, in range $(0, 2N)$ of Δ_Σ (acting on functions).

Geometric Significance: Gives the Morse Index of Σ as a critical point of volume, but restricted to Legendrian variations.

Prop^n: (Haskins, Joyce) If C is a SLG cone with isolated singularity then l-ind$(C) \geq 2N$.

Proof: Σ^{N-1} minimal submanifold in $\mathbb{R}^{2N}$
$\Rightarrow$ coordinate functions $x_1, \ldots, x_{2N}$ give e-functions of Δ with e-value $N-1$.

Then prove that isolated SLG singularity
$\Rightarrow \Sigma^{N-1}$ is linearly full
$\Rightarrow x_1, \ldots, x_{2N}$ are independent
$\Rightarrow l$-ind $\geq 2N$.

(Not linearly full
$\Rightarrow C$ has some translation invariance
$\Rightarrow C$ doesn't have isolated singularity.)

Remarks about l-index and 2nd variation

Remarks:

1. Round spheres/planes contribute exactly N to l-ind(C).

2. SU(N) acts on SLG cones preserving the SLG condition.

$\Rightarrow$ Certain quadratic functions on $\mathbb{C}^N$ restrict to give e-functions $\Delta f = 2Nf$.

e.g., $|z_i|^2 - |z_j|^2$, Re$(z_i\overline{z_j})$, Im$(z_i\overline{z_j})$
(they are all Hamiltonian functions for SU(N) vector fields).

3. For any SLG T^2-cone C can prove (Haskins 2002) that l-ind(C) = 6 iff C is the cone over the Clifford torus. (Higher diml analogue is false).

Def^n: C is **rigid** if the only infinitesimal deformations arise from the action of SU(N).

Spectral curve genus and second variation operator

Fundamental fact: any minimal Lagrangian 2-torus in $\mathbb{C}P^2$ with spectral curve of genus $g = 2d$ comes in at least a real $d-2$-dimensional family of non-congruent minimal Lagrangian tori.

(d-diml family arises by moving line bundle $\mathcal{L}$ inside the Prym variety of the spectral curve. -2 occurs by factoring choice of basepoint in T^2).

Fundamental fact $\Rightarrow$ a SLG T^2-cone of spectral curve genus $g = 2d$ has $\lambda = 6$ as a eigenvalue of Δ_Σ with multiplicity m

$$\text{m} \geq \dim(SU(3)) + (d-2) = 6 + d.$$

Legendrian index and spectral curve genus

Thm (Haskins 2003): Let C be a special Lagrangian T^2-cone in $\mathbb{C}^3$ of spectral curve genus $2d$. Then for any $d \geq 3$,

$$\text{l-ind}(C) \geq \max\{\left[\frac{1}{2}d\right], 7\}.$$

Corollary: There exist special Lagrangian T^2-cones with arbitrarily large l-index.

Proof use following result from spectral geometry of Δ on 2-tori. (have analogues for higher genus but not higher dimensions).

Thm (Nadirashvili 1987) Let (Σ, h) be an Riemannian 2-torus, and let m_i be the multiplicity of the i-th eigenvalue of Δ_Σ acting on functions. Then

$$m_i \leq 2i + 4.$$

The heat kernel and minimal submanifolds of S^{n+l}

On Riemannian mfd (M,g) the heat kernel $H(x,y,t)$ is the fundamental solution to the heat equation on M. M cpt implies trace of the heat kernel can be written in terms of eigenvalues λ_i of Δ_g

$$\int_M H(x,x,t) = \sum_{i=0}^{\infty} e^{-\lambda_i t}.$$

Cheng-Li-Yau proved heat kernel comparison results for minimal submanifolds of space forms.

Thm (C-L-Y 1984) Let M be a cpt n-mfd with no bdy minimally immersed in S^{n+l}. Let λ_i denote evals of Δ on M and on S^n respectively. Then

$$\sum_{i=0}^{\infty} e^{-\lambda_i t} \leq \Theta(M) \sum_{i=0}^{\infty} e^{-\mu_i t}$$

holds for all $t > 0$, where $\Theta(M) = Vol(M)/Vol(S^n)$.

The heat kernel and minimal submanifolds of S^{n+l}

Apply this result to the link Σ of a SLG T^2-cone C

$$\begin{aligned}\sum_{i=0}^{\infty} e^{-\lambda_i t} &> 1 + (\text{l-ind}(C) + \text{l-nullity}(C))e^{-6t} \\ &> 1 + \left(\frac{d}{2} + (6+d)\right) e^{-6t}. \quad (*)\end{aligned}$$

Also eigenvalues μ_i of S^2 and their multiplicities are known. i-th distinct eigenvalue is $i(i+1)$ with multiplicity $2i+1$. This easily gives an upper bound for $\sum e^{-\mu_i t}$.

$$\text{Area}(\Sigma) > \frac{1}{3} d\pi$$

follows by combining these two inequalities with the Cheng-Li-Yau heat kernel inequality and choosing a value of t to give reasonable constants.