MANIFOLDS WITH SPECIAL HOLONOMY
LECTURE 3:
HYPERKÄHLER AND QUATERNION-KÄHLER

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Now, I want to move down the list a bit, using what we know about the Calabi-Yau case. I'll skip the $\mathfrak{sp}(m) \oplus \mathfrak{sp}(1)$ case and come back to it.

n	$\mathfrak{h} \subseteq \mathfrak{so}(n)$	$K(\mathfrak{h})$ as an $\mathfrak{h}$-module
n	$\mathfrak{so}(n)$	$\mathbb{R} \oplus \mathsf{S}^2_0(\mathbb{R}^n) \oplus W_n(\mathbb{R}^n)$
$n = 2m > 2$	$\mathfrak{u}(m)$	$\mathbb{R} \oplus \mathsf{S}^{1,1}_0(\mathbb{C}^m)^{\mathbb{R}} \oplus \mathsf{S}^{2,2}_0(\mathbb{C}^m)^{\mathbb{R}}$
$n = 2m > 2$	$\mathfrak{su}(m)$	$\mathsf{S}^{2,2}_0(\mathbb{C}^m)^{\mathbb{R}}$
$n = 4m > 4$	$\mathfrak{sp}(m) \oplus \mathfrak{sp}(1)$	$\mathbb{R} \oplus \mathsf{S}^4(\mathbb{C}^{2m})^{\mathbb{R}}$
$n = 4m > 4$	$\mathfrak{sp}(m)$	$\mathsf{S}^4(\mathbb{C}^{2m})^{\mathbb{R}}$
$n = 7$	$\mathfrak{g}_2$	$\mathsf{V}^{0,2} \simeq \mathbb{R}^{77}$
$n = 8$	$\mathfrak{spin}(7)$	$\mathsf{V}^{0,2,0} \simeq \mathbb{R}^{168}$

3. Symplectic Unitary Holonomy. The subgroup $\mathrm{Sp}(m) \subset \mathrm{SO}(4m)$ is defined as the subgroup of $\mathrm{GL}(4m, \mathbb{R})$ that preserves the pair of forms on $\mathbb{C}^{2m} = \mathbb{R}^{4m}$ defined as

$$\omega_0 = \tfrac{1}{2}\,\mathrm{i}\left(\mathrm{d}z^1 \wedge \mathrm{d}\overline{z^1} + \cdots + \mathrm{d}z^{2m} \wedge \mathrm{d}\overline{z^{2m}}\right)$$
$$\Omega_0 = \mathrm{d}z^1 \wedge \mathrm{d}z^2 + \mathrm{d}z^3 \wedge \mathrm{d}z^4 + \cdots + \mathrm{d}z^{2m-1} \wedge \mathrm{d}z^{2m}.$$

Since

$${\Omega_0}^m = m!\,\mathrm{d}z^1 \wedge \cdots \mathrm{d}z^{2n} = m!\,\Upsilon_0\,,$$

and since $\mathrm{SU}(2m)$ is the subgroup of $\mathrm{GL}(4m, \mathbb{R})$ that preserves the pair (ω_0, Υ_0) on $\mathbb{C}^{2m} = \mathbb{R}^{4m}$, $\mathrm{Sp}(m)$ is a subgroup of $\mathrm{SU}(2m)$.

Though $\mathrm{Sp}(1) = \mathrm{SU}(2)$, the group $\mathrm{Sp}(m)$ is a proper subgroup of $\mathrm{SU}(2m)$ when $m > 1$. In fact, $\mathrm{Sp}(m)$ is a compact simple Lie group and a maximal compact subgroup of $\mathrm{Sp}(m, \mathbb{C})$, the subgroup of $\mathrm{GL}(4m, \mathbb{R})$ consisting of the linear transformations that fix Ω_0.

The rank of $\mathrm{Sp}(m)$ is m and its dimension is $2m^2 + m$. It acts irreducibly on $\mathbb{R}^{4m} = \mathbb{C}^{2m}$ and, moreover, transitively on the unit sphere S^{4m-1}.

Quaternionic Structures There is another interpretation (or definition) of $\mathrm{Sp}(m)$ that may be more enlightening. For indicial symmetry, write

$$\omega_1 = \omega, \qquad \omega_2 = \mathrm{Re}(\Omega), \qquad \omega_3 = \mathrm{Im}(\Omega),$$

and note that each ω_i is a nondegenerate 2-form on $\mathbb{R}^{4m}$. This implies that there are maps $J_i : \mathbb{R}^{4m} \to \mathbb{R}^{4m}$ for $i = 1, 2, 3$ so that

$$\begin{aligned} \omega_1(J_2 x, y) &= \omega_3(x, y), \\ \omega_2(J_3 x, y) &= \omega_1(x, y), \\ \omega_3(J_1 x, y) &= \omega_2(x, y). \end{aligned}$$

You can check that ${J_i}^2 = -1$ and $J_i J_j = -J_j J_i = -J_k$ whenever (i, j, k) is an even permutation of $(1, 2, 3)$. (Also, $J_i \in \mathrm{SO}(4m)$.) Moreover, the associated inner product g satisfies

$$g(x, y) = \omega_i(x, J_i y) \qquad \text{for } i = 1,\, 2,\, 3.$$

So, thinking of $\mathbb{R}^{4m}$ as $\mathbb{H}^m$, we see that $\mathrm{Sp}(m)$ is the set of $\mathbb{H}$-linear orthogonal transformations of $\mathbb{H}^m$.

Definition: A triple of nondegenerate 3-forms (η_1, η_2, η_3) on a real vector space V will be said to be a *hyper-unitary structure* on V if the equations

$$\eta_1(J_2 x, y) = \eta_3(x, y),$$
$$\eta_2(J_3 x, y) = \eta_1(x, y),$$
$$\eta_3(J_1 x, y) = \eta_2(x, y).$$

define linear maps $J_i : V \to V$ that satisfy ${J_i}^2 = -1$ and that $J_i J_j = -J_j J_i = -J_k$ whenever (i, j, k) is an even permutation of $(1, 2, 3)$ and, if, moreover, the expressions

$$\eta_1(x, J_1 y) = \eta_2(x, J_2 y) = \eta_3(x, J_3 y)$$

all agree and define a positive definite inner product $\langle , \rangle$ on V.

Proposition: V has a hyper-unitary structure if and only if $\dim V$ is divisible by 4. Moreover, all hyper-unitary structures on a vector space V are isomorphic.

Hyper-Kähler structures. Suppose that (M^{4m}, g) is a Riemannian manifold whose holonomy is conjugate to a subgroup of $\mathrm{Sp}(m)$.

By the holonomy principle, there will be three nondegenerate 2-forms, say $(\omega_1, \omega_2, \omega_3)$ on M that are parallel with respect to g such that, at each point $x \in M$, they define a hyper-unitary structure on $T_x M$.

Of course, these 2-forms are closed and, in fact, (M, g, J_i, ω_i) is Kähler for $i = 1$, 2, and 3!

In fact, even more is true: For any constants $(\lambda_1, \lambda_2, \lambda_3)$ such that ${\lambda_1}^2+{\lambda_2}^2+{\lambda_3}^2=1$, the data $(M, g, J_\lambda, \omega_\lambda)$ is Kähler, where

$$J_\lambda = \lambda_1 J_1+\lambda_2 J_2+\lambda_3 J_3 \qquad \text{and} \qquad \omega_\lambda = \lambda_1\omega_1+\lambda_2\omega_2+\lambda_3\omega_3\,.$$

(We often say that g is hyper-Kähler.)

By now, the following ‘converse’ should not be surprising:

Theorem: Suppose that a manifold M has a triple of closed 2-forms $(\omega_1, \omega_2, \omega_3)$ such that, at each point $x \in M$, they define a hyper-unitary structure on T_xM. Then these three forms are parallel with respect to the associated metric g. (Whose holonomy is therefore conjugate to a subgroup of $\mathrm{Sp}(m)$ where $\dim M = 4m$.)

(As usual, the proof will indicate how to construct such examples, at least locally.)

Sketch of proof: By the algebraic properties of hyper-unitary triples, we know that $\dim M = 4m$ for some integer m.

Moreover, setting $\Omega = \omega_2 + \mathrm{i}\,\omega_3$, we see that the complex-valued $2m$-form $\Upsilon = \frac{1}{m!}\Omega^m$ is decomposable as a complex valued form, satisfies $\Upsilon \wedge \overline{\Upsilon} \neq 0$ and is closed.

By a previous argument, we know that Υ is a holomorphic volume form on M for a unique (integrable) complex structure J (which happens to equal J_1).

Next, Ω itself is of type $(2,0)$ with respect to this complex structure J and closed, so it is holomorphic with respect to the underlying complex structure.

We can now apply the holomorphic Darboux theorem to see that each point of M lies in a neighborhood U on which there exist coordinates $z : U \to \mathbb{C}^{2m}$ such that

$$\Omega_U = \mathrm{d}z^1 \wedge \mathrm{d}z^{m+1} + \mathrm{d}z^2 \wedge \mathrm{d}z^{m+2} + \cdots + \mathrm{d}z^m \wedge \mathrm{d}z^{2m}.$$

We've also seen that (M, g, J_1, ω_1) is Kähler, so locally, there is a function f on U so that

$$U^*(\omega_1) = \frac{\mathrm{i}}{2} \frac{\partial^2 f}{\partial z^j \partial \overline{z^k}} \mathrm{d}z^j \wedge \mathrm{d}\overline{z^k} \qquad \text{and} \qquad U^*(g) = \frac{\partial^2 f}{\partial z^j \partial \overline{z^k}} \mathrm{d}z^j \circ \mathrm{d}\overline{z^k}$$

where

$$H_f = \left(\frac{\partial^2 f}{\partial z^j \partial \overline{z^k}} \right) > 0.$$

is the complex Hessian matrix of the function f.

Finally, the condition that $(\omega_1, \omega_2, \omega_3)$ define a hyper-unitary structure at each point imposes many more equations on f than just the Monge-Ampere equation. In fact, this becomes the system of equations:

$$ {}^t H_f \begin{pmatrix} 0_m & \mathrm{I}_m \\ -\mathrm{I}_m & 0_m \end{pmatrix} H_f = \begin{pmatrix} 0_m & \mathrm{I}_m \\ -\mathrm{I}_m & 0_m \end{pmatrix}. $$

This turns out to be $2m^2 - m$ second order nonlinear equations on f. (Taking determinants of both sides of this equation gives the Monge-Ampere equation that defines $\mathrm{SU}(2m)$-holonomy metrics.)

Conversely, if f on $U \subset \mathbb{C}^{2m}$ satisfies this system of equations and has positive definite Hessian H_f, then the metric g defined on U by

$$ g = \frac{\partial^2 f}{\partial z^j \, \partial \overline{z^k}} \, \mathrm{d}z^j \circ \mathrm{d}\overline{z^k} $$

will have ω_1, and $\Omega = \omega_2 + \mathrm{i}\,\omega_3$ as parallel 2-forms on U, so its holonomy will be a subgroup of $\mathrm{Sp}(m)$.

Local Properties of HyperKähler: $(M^{4m}, \omega_1, \omega_2, \omega_3)$.

(1) Involutive PDE analysis implies that the general solution of the Hessian equation depends on $2m$ 'arbitrary' functions of $2m+1$ (real) variables.
(2) The 'generic' solution has holonomy equal to $\mathrm{Sp}(m)$.
(3) The associated Riemannian manifold (M, g) is Ricci-flat.
(4) The associated Riemannian manifold (M, g) supports many different calibrations, e.g., $\phi_p(\lambda) = \frac{1}{p!}{\omega_\lambda}^p$ and the real part of Υ_λ, and all of these calibrate many submanifolds of (M, g).
(5) The structure $(M, \Omega) = (M, \omega_2 + \mathrm{i}\,\omega_3)$ is a holomorphic symplectic manifold.

Global Examples:

Calabi's Example: Looking for a rotationally invariant example on $\mathbb{C}^{2m}$ won't work (we already know that you'll get $\mathrm{SU}(2m)$-holonomy anyway).

However, there is a natural holomorphic symplectic manifold that does have a high degree of symmetry: $X = T^*(\mathbb{CP}^m)$.

The group $\mathrm{SU}(m{+}1)$ acts on $T^*(\mathbb{CP}^m)$ and (because $\mathbb{CP}^m$ is a rank one symmetric space), its general orbit is a (real) hypersurface in $T^*(\mathbb{CP}^m)$ the level sets of the Hermitian norm function $\rho : T^*(\mathbb{CP}^m) \to \mathbb{R}$.

Calabi's Idea: Look for a hyperKähler structure on $T^*(\mathbb{CP}^m)$ of the form $(\omega_1, \omega_2, \omega_3)$ where $\Omega = \omega_2 + \mathrm{i}\,\omega_3$ is the canonical holomorphic symplectic structure on $T^*(\mathbb{CP}^m)$ and where

$$\omega_1 = \mathrm{i}\partial\bar{\partial}\big(f(\rho)\big)$$

for some function f of one variable (defined by some ODE).

Result: This works. There is such a function f and the resulting structure is complete on $T^*(\mathbb{CP}^m)$.

Compact examples. A hyperKähler structure $(M, \omega_1, \omega_2, \omega_3)$ is a Kähler manifold (M, g, J_1, ω_1) that has a parallel holomorphic symplectic form $\Omega = \omega_2 + \mathrm{i}\,\omega_3$.

It is not difficult to show that if ϕ is any holomorphic p-form on M and M is compact, then ϕ is also g-parallel:

$$\nabla^*\nabla\phi = \bar{\partial}^*\bar{\partial}\phi + \mathrm{Ric}\cdot\phi,$$

but $\mathrm{Ric}(g) = 0$ since g has holonomy in $\mathrm{Sp}(m) \subset \mathrm{SU}(m)$.

If the holonomy of g is to be all of $\mathrm{Sp}(m)$, then the only holomorphic differential forms on M are the powers of Ω.

Proposition: Let M^{4m} be a simply-connected, compact complex manifold that admits a Kähler metric and whose algebra of holomorphic forms is generated by a holomorphic symplectic form $\Omega = \omega_2 + \mathrm{i}\,\omega_3$. Then M supports a Kähler structure (g, J_1, ω_1) such that $(M, \omega_1, \omega_2, \omega_3)$ is hyperKähler and g has holonomy $\mathrm{Sp}(m)$.

Proof: Apply Yau's theorem to M with the holomorphic volume form $\Upsilon = \frac{1}{m!}\Omega^m$ and some Kähler form $\omega_0 \in \Omega^{1,1}_+(M)$.

Thus, let (ω_1, Υ) be the Calabi-Yau structure on M where

$$\omega_1 = \lambda\omega_0 + \mathrm{i}\partial\bar{\partial} f.$$

for some constant $\lambda > 0$ and function f on M.

By argument above, Ω is g-parallel for the underlying metric g of (ω_1, Υ), so the holonomy of g is a subgroup of $\mathrm{Sp}(m)$.

If the holonomy acts irreducibly, then by Berger's classification, it must be $\mathrm{Sp}(m)$.

If the holonomy were to act reducibly, then by de Rham and Berger, the holonomy would be of the form

$$\mathrm{Sp}(m_1) \times \mathrm{Sp}(m_2) \times \cdots \times \mathrm{Sp}(m_k)$$

for some $m_1 + m_2 + \cdots + m_k = m$. But, if $k > 1$, we could write

$$\Omega = \Omega_1 + \cdots + \Omega_k$$

where the Ω_i are nonzero holomorphic 2-forms. By hypothesis, the only holomorphic 2-forms on M are the multiples of Ω, so $k = 1$.

Explicit examples: Finally, several methods of constructing simply-connected, compact Kähler manifolds whose holomorphic forms are generated by a holomorphic symplectic form are known via algebraic geometry.

The first and most famous example is to start with a K3 surface X_2, take a symmetric product

$$Y_m = {X_2}^{(m)} = \big(X_2 \times X_2 \times \cdots \times X_2\big)/S_m$$

and then resolve the singularities of Y_m in a nice way, getting the desired manifold X_m.

The holomorphic volume forms on the factors pull up to the product and add to give a symplectic 2-form Ω that survives through the quotient and resolution to define a symplectic 2-form on X_m.

Now need to go back to pick up the "quaternionic-Kähler" case.

Dimension	Group	Invariant forms (generators)
n	$\mathrm{SO}(n)$	$1 \in \Lambda^0,\ *1 \in \Lambda^n$
$n = 2m$	$\mathrm{U}(m)$	$1 \in \Lambda^0,\ \omega \in \Lambda^2$
$n = 2m$	$\mathrm{SU}(m)$	$1 \in \Lambda^0,\ \omega \in \Lambda^2,\ \phi, \psi \in \Lambda^m$
$n = 4m$	$\mathrm{Sp}(m){\cdot}\mathrm{Sp}(1)$	$1 \in \Lambda^0,\ \Phi \in \Lambda^4$
$n = 4m$	$\mathrm{Sp}(m)$	$1 \in \Lambda^0,\ \omega_1, \omega_2, \omega_3 \in \Lambda^2$
$n = 7$	G_2	$1 \in \Lambda^0,\ \phi \in \Lambda^3,\ *\phi \in \Lambda^4$
$n = 8$	$\mathrm{Spin}(7)$	$1 \in \Lambda^0,\ \Phi \in \Lambda^4$

4. Quaternionic-Kähler Holonomy: Since $J_1, J_2, J_3 \in \mathrm{SO}(4m)$ don't commute, they don't belong to $\mathrm{Sp}(m) \subset \mathrm{SO}(4m)$. Instead, they generate a group isomorphic to $\mathrm{Sp}(1)$

$$\mathrm{Sp}(1) = \left\{ \lambda_0 \mathrm{I}_{4m} + \lambda_1\, J_1 + \lambda_2\, J_2 + \lambda_3\, J_3 \,\middle|\, {\lambda_0}^2 + {\lambda_1}^2 + {\lambda_2}^2 + {\lambda_3}^2 = 1 \right\}$$

that commutes with $\mathrm{Sp}(m) \subset \mathrm{SO}(4m)$.

The group jointly generated by $\mathrm{Sp}(m)$ and this $\mathrm{Sp}(1)$ is denoted $\mathrm{Sp}(m){\cdot}\mathrm{Sp}(1) \subset \mathrm{SO}(4m)$. This group does not leave the 2-forms ω_i invariant, but does leave the 4-form

$$\Phi_0 = \frac{1}{6}\left({\omega_1}^2 + {\omega_2}^2 + {\omega_3}^2\right)$$

invariant. (The $\frac{1}{6}$ is chosen to make Φ_0 have comass 1.)

Conversely, when $m > 1$, the group $\mathrm{Sp}(m){\cdot}\mathrm{Sp}(1)$ can be *defined* as the subgroup of $\mathrm{GL}(4m, \mathbb{R})$ that fixes Φ_0.

If V is a vector space of dimension $4m$, a form $\Psi \in \Lambda^4(V^*)$ will be said to be a *quaternionic* 4-form if it is equivalent to Φ_0 under some linear isomorphism $V \to \mathbb{R}^{4m}$.

n	$\mathfrak{h} \subseteq \mathfrak{so}(n)$	$K(\mathfrak{h})$ as an $\mathfrak{h}$-module
n	$\mathfrak{so}(n)$	$\mathbb{R} \oplus \mathsf{S}^2_0(\mathbb{R}^n) \oplus W_n(\mathbb{R}^n)$
$n = 2m > 2$	$\mathfrak{u}(m)$	$\mathbb{R} \oplus \mathsf{S}^{1,1}_0(\mathbb{C}^m)^{\mathbb{R}} \oplus \mathsf{S}^{2,2}_0(\mathbb{C}^m)^{\mathbb{R}}$
$n = 2m > 2$	$\mathfrak{su}(m)$	$\mathsf{S}^{2,2}_0(\mathbb{C}^m)^{\mathbb{R}}$
$n = 4m > 4$	$\mathfrak{sp}(m) \oplus \mathfrak{sp}(1)$	$\mathbb{R} \oplus \mathsf{S}^4(\mathbb{C}^{2m})^{\mathbb{R}}$
$n = 4m > 4$	$\mathfrak{sp}(m)$	$\mathsf{S}^4(\mathbb{C}^{2m})^{\mathbb{R}}$
$n = 7$	$\mathfrak{g}_2$	$\mathsf{V}^{0,2} \simeq \mathbb{R}^{77}$
$n = 8$	$\mathfrak{spin}(7)$	$\mathsf{V}^{0,2,0} \simeq \mathbb{R}^{168}$

Local Properties of Quaternionic-Kähler manifolds: Let (M^{4m}, Φ) $(m > 1)$ be a manifold endowed with a quaternionic form $\Phi \in \Omega^4(M)$ (i.e., Φ_x is quaternionic on T_xM for all $x \in M$). Let g be the associated metric (assumed not locally symmetric).

(1) If Φ is g-parallel, then g is Einstein.
(2) If $\mathrm{Scal}(g) \neq 0$, then holonomy of g is $\mathrm{Sp}(m){\cdot}\mathrm{Sp}(1)$.
(3) If $\mathrm{Scal}(g) = 0$, then holonomy of g lies in $\mathrm{Sp}(m)$.
(4) When $m > 2$, $\mathrm{d}\Phi = 0$ implies that Φ is g-parallel.
(5) When g has holonomy $\mathrm{Sp}(m){\cdot}\mathrm{Sp}(1)$, the form Φ can be constructed via 'reduction' from a hyperKähler structure $(N^{4m+4}, \omega_1, \omega_2, \omega_3)$ with an S^1-symmetry.
(6) The general (local) metric g with holonomy $\mathrm{Sp}(m){\cdot}\mathrm{Sp}(1)$ depends on $2m$ functions of $2m{+}1$ (real) variables.
(7) All compact examples with $\mathrm{Scal} > 0$ with $m = 2$ or 3 are Riemannian symmetric spaces.
(8) No compact examples with $\mathrm{Scal} > 0$ are known that are not Riemannian symmetric spaces.