Semidefinite relaxations for $0/1$ Polytopes

Monique LAURENT

CWI, Amsterdam

A guiding example: The Max-Cut problem

$$\begin{array}{ll} \max & \frac{1}{2}\sum_{ij\in E} w_{ij}(1-x_ix_j) \\ \text{s.t.} & x\in\{\pm 1\}^n \end{array}$$

- Max-Cut is **NP-hard** [Karp 1972]

- Max-Cut is **polynomial** for graphs with no K_5 minor [Barahona-Mahjoub 1986]

$\rightsquigarrow$ use the **LP relaxation:**

$$\begin{array}{lll} \max & \frac{1}{2}\sum_{ij\in E} w_{ij}(1-x_{ij}) & \\ \text{s.t.} & x_{ij}+x_{ik}+x_{jk}\geq -1 & \\ & x_{ij}-x_{ik}-x_{jk}\geq -1 & (i,j,k\in[1,n]) \end{array}$$

- Max-Cut has a 0.878**-approximation algorithm** [Goemans-Williamson 1995]

$\rightsquigarrow$ use the **SDP relaxation:**

$$\begin{array}{ll} \max & \frac{1}{2}\sum_{ij\in E} w_{ij}(1-x_{ij}) \\ \text{s.t.} & X=(x_{ij})\succeq 0 \\ & \text{diag}(X)=1 \end{array}$$

How to define stronger SDP relaxations?

• Add valid linear inequalities *explicitely* to the basic SDP relaxation; e.g., triangle inequalities.

$\rightsquigarrow$ 0.932-approximation algorithm for Max-Cut in cubic graphs [Halperin Livnat Zwick 2002]

• Use SDP relaxations containing *implicitely* strong valid linear inequalities

1. Iterative 'matrix-cut' method [Lovász-Schrijver 1991]
2. Taking Lagrangian bi-dual [Poljak-Rendl-Wolkowicz 1995, Anjos-Wolkowicz 2000]
3. Real-algebraic method [Shor 1987, Nesterov 1997, Lasserre, Parrilo 2000]

Taking the dual of the Lagrangian dual (the bidual) [Poljak Rendl Wolkowicz 1995] [Shor 1985] .... [Lemaréchal Oustry 2000]

Example of max-cut:

$$\max x^T Q x \;\; \text{subject to} \;\; x_i^2 = 1 \; (i = 1, \ldots, n)$$

Lagragian dual:

$$\min_{u \in \Re^n} \max_{x \in \Re^n} x^T Q x + \sum_i u_i (1 - x_i^2)$$

$$\min_{u \in \Re^n} \max_{x \in \Re^n} x^T (Q - \text{diag} u) x + u^T e$$

$$\begin{cases} \max = +\infty \text{ if } Q - \text{diag} u \not\preceq 0 \\ \max = u^T e \;\; \text{if } Q - \text{diag} u \preceq 0 \end{cases}$$

$$= \min u^T e \;\; \text{subject to} \;\; \text{diag} u - Q \succeq 0$$

Taking the dual:

$$= \max \langle Q, X \rangle \;\; \text{subject to diag} X = e, \; X \succeq 0$$

$\rightsquigarrow$ basic SDP relaxation

Idea: get stronger relaxations by adding redundant constraints to the max-cut formulation [Anjos Wolkowicz 2000]

max-cut:

$$\begin{array}{ll} \max & \langle Q, X\rangle \\ \text{s.t.} & X_{ii} = 1 \ (i = 1, \ldots, n) \\ & X \succeq 0 \\ & \operatorname{rank}(X) = 1 \\ & X_{ij}^2 = 1 \ (i \neq j = 1, \ldots, n) \\ & X^2 - nX = 0 \\ & X_{ij} = X_{ik} X_{jk} \ (i \neq j \neq k = 1, \ldots, n) \end{array}$$

Bidual:

$$\begin{array}{ll} \max & \sum_{ij \in E_n} Q_{ij} Y_{0,ij} \\ \text{s.t.} & Y \succeq 0 \\ & \operatorname{diag} Y = e \\ & Y_{ik,jk} = Y_{0,ij} \ (i \neq j \neq k = 1 \ldots n) \end{array}$$

Fact: $N_+(\text{MET}_n)$ is contained in the AW relaxation

A general lifting paradigm for finding conv(F) where $F \subseteq \{\pm 1\}^n$

$$x \in F \rightsquigarrow y := (\prod_{i \in I} x_i)_{I \subseteq V} \rightsquigarrow Y := yy^T \text{ satisfies:}$$

(i) $\text{diag}(Y) = 1$
(ii) $Y(I, J)$ depends only on $I \Delta J$
(iii) $Y \succeq 0$
(iv) + conditions depending on F

As Y is indexed by all subsets of $V = \{1, \ldots, n\}$, it is exponentially big
$\rightsquigarrow$ restrict to submatrices of Y of polynomial sizes

- *Lovász-Schrijver:* Restrict to the principal submatrix indexed by $\emptyset$ and the singletons $1, \ldots, n$
- *Lasserre:* Restrict to the principal submatrix indexed by all subsets of size $\leq t$
- *Sherali-Adams:* Restrict to the principal submatrices indexed by all subsets of U for $U \subseteq V$ with $|U| = t$

The methods also differ in the way of expressing membership in conv(F)

The Lovász-Schrijver construction

$P :=$ CUT(G) in the edge space $\Re^E$
$K :=$ MET(G) linear relaxation of P
(Generally: $K \subseteq [-1,1]^d$ convex, $P :=$ conv $(K \cap \{\pm 1\}^d)$)

$$z \in K \cap \{\pm 1\}^E \rightsquigarrow Z := \begin{pmatrix} 1 \\ z \end{pmatrix} (1 \; z^T) \text{ satisfies:}$$

$$\boxed{\begin{array}{ll} \text{(i)} & \text{diag}(Z) = 1 \\ \text{(ii)} & Z(e_0 \pm e_{ij}) \in \tilde{K} \;\; (ij \in E) \\ \text{(iii)} & Z \succeq 0 \end{array}}$$

$$N(K) := \{z \in \Re^E \mid \begin{pmatrix} 1 \\ z \end{pmatrix} = Ze_0 \text{ for } Z \text{ satisfying } (i)-(ii)\}$$

$$N_+(K) := \{z \in \Re^E \mid \begin{pmatrix} 1 \\ z \end{pmatrix} = Ze_0 \text{ for } Z \text{ satisfying } (i)-(iii)\}$$

$$\boxed{P \subseteq N_+(K) \subseteq N(K) \subseteq K}$$

Iterated relaxations:

$$\boxed{\begin{array}{c} N^t(K) := N(N^{t-1}(K)) \\ N_+^t(K) := N_+(N_+^{t-1}(K)) \end{array}}$$

Facts:

- One can optimize in polynomial time over $N^t(K)$, $N_+^t(K)$ for fixed t assuming existence of an efficient separation algorithm for K

- $N^d(K) = P$ where $d = dim(K)$ $(= |E|$ here)

- For $K = \mathrm{MET}(G)$, $P = \mathrm{CUT}(G)$
$G/\{e_1, \ldots, e_t\}$ has no K_5-minor $\Longrightarrow N^t(K) = P$

- For $K = \mathrm{FR}(G)$, $P = \mathrm{STAB}(G)$
$N^{n-\alpha(G)-1}(K) = N_+^{\alpha(G)}(K) = P$

The Lasserre construction

$x \in \{\pm 1\}^n \rightsquigarrow y := (\prod_{i \in I} x_i)_{I \subseteq V} \rightsquigarrow Y := yy^T$ satisfies:

$$\boxed{\begin{array}{l} Y \succeq 0 \\ \text{diag}(Y)=1 \\ Y(I,J) \text{ depends only on } I\Delta J \end{array}}$$

Y is a moment matrix

Definition: Given an integer $t \geq 1$ and a vector $y = (y_I)_{|I| \leq 2t}$, its *moment matrix* $M_t(y)$ of *order* t is

$$\boxed{M_t(y) := (y_{I\Delta J})_{|I|,|J| \leq t}}$$

$Q_t(G) :=$ projection on $\Re^E$ of set $\{y \mid M_t(y) \succeq 0,\ y_\emptyset = 1\}$

$$\text{CUT}(G) \subseteq Q_n(G) \subseteq \ldots \subseteq Q_t(G) \subseteq \ldots \subseteq Q_1(G)$$

Lemma: The eigenvectors of $M_n(y)$ are the vectors $y^A = ((-1)^{|A \cap I|})_{I \subseteq V}$ with eigenvalue $y^T y^A$. That is,

$$M_n(y) = \sum_{A \subseteq V} \frac{y^T y^A}{2^n}\, y^A (y^A)^T$$

$$\boxed{\text{CUT}(G) = Q_n(G)}$$

Intermezzo: The Lasserre construction for general ± 1 polytopes

$$P = \text{conv-hull}\ (K \cap \{\pm 1\}^n)$$

$K \subseteq [-1,1]^n$ polytope or semi-algebraic set

$$K = \{x \in \Re^n \mid g_1(x) \geq 0 \ldots g_m(x) \geq 0\}$$

may assume $g(x) = \sum_{I \subseteq V} g_I \prod_{i \in I} x_i$

$$\boxed{g, y \in \Re^{\mathcal{P}(V)} \rightsquigarrow g * y := M_n(y) g}$$

Observation: $x \in K \cap \{\pm 1\}^n \rightsquigarrow y := (\prod_{i \in I} x_i)_{I \subseteq V}$ satisfies: $g * y = g(x) \cdot y$ and, therefore,

$$\boxed{\begin{array}{l} M_n(y) \succeq 0 \\ M_n(g_\ell * y) \succeq 0 \quad \forall \ell = 1, \ldots, m \end{array}}$$

$Q_t(K) :=$ projection on $\Re^n$ of the set

$$\{y \mid y_\emptyset = 1,\ M_t(y) \succeq 0,\ M_{t-v_\ell}(g_\ell * y) \succeq 0\ \forall \ell\}$$

for $t \geq v := \max \left\lceil \frac{deg(g_\ell)}{2} \right\rceil$

$$P \subseteq Q_{n+v}(K) \subseteq \ldots \subseteq Q_v(K)$$

$$\boxed{P = Q_{n+v}(K)}$$

$$\boxed{P \stackrel{?}{=} Q_{n+v}(K)}$$

Assume:

$M_n(y) \succeq 0$, $M_n(g_\ell * y) \succeq 0 \ \forall \ell$

Then:

$$y = \sum_A \lambda_A \, y^A$$

with $\lambda_A := \frac{y^T y^A}{2^n} \geq 0$ for all A

Show: $\lambda_A = 0$ if A does not correspond to a point in K; that is, if $g_\ell^T y^A < 0$ for some ℓ

Proof: The eigenvalue of $M_n(g_\ell * y)$ for eigenvector y^A is equal to:

$$\begin{aligned}(g_\ell * y)^T y^A &= \sum_I (g_\ell * y)_I \ y_I^A \\ &= \sum_I \left(\sum_J g_\ell(J) y_{I \Delta J} \right) y_I^A \\ &= \sum_J g_\ell(J) y_J^A \left(\sum_I y_{I \Delta J} y_{I \Delta J}^A \right) \\ &= g_\ell^T y^A \cdot y^T y^A \geq 0\end{aligned}$$

Hence:

$$\begin{cases} y^T y^A \geq 0 \\ g_\ell^T y^A < 0 \end{cases} \Longrightarrow y^T y^A = 0$$

Comparison of the Lasserre and Lovász-Schrijver constructions

K is a polytope

$$\boxed{\begin{array}{c} Q_{t+1}(K) \subseteq N_+(Q_t(K)) \\ \Downarrow \\ Q_{t+1}(K) \subseteq N_+^t(K) \end{array}}$$

Example: $G = (V, E)$ graph
K = fractional stable set polytope FR(G)

$$= \{x \in \Re_+^V \mid x_i + x_j \leq 1 \ (ij \in E)\}$$

Then:
$Q_t(K)$ = projection on $\Re^V$ of the set

$$\{y \mid y_\emptyset = 1, \ M_t(y) \succeq 0, \ y_{ij} = 0 \ (ij \in E)\}$$

Hence:

$$Q_2(K) \subseteq N_+(K) \subseteq \mathrm{TH}(G) = Q_1(G)$$

$$\mathrm{STAB}(G) = Q_\alpha(K) \subseteq N_+^{\alpha-1}(K)$$

strict inclusion for the line graph of K_{2n+1}
(Stephen-Tunçel 1999)

Comparison with the Sherali-Adams lift-and-project method

$$K = \{x \in [-1,1]^n \mid g_1(x) \geq 0 \ldots g_m(x) \geq 0\}$$

(1) Multiply each inequality defining K by the products $\prod_{i\in A}(1-x_i) \prod_{i\in U\setminus A}(1+x_i)$ for all $A \subseteq U \subseteq [1,n]$ with $|U| = t$

(2) Linearize: $y_I := \prod_{i\in I} x_i$, $x_i^2 = 1 \;\; \forall i$

(3) $S_t(K) :=$ projection on the x-space $\Re^n$

$$\boxed{S_t(K) \subseteq N^t(K)}$$

Interpretation in terms of moment matrices:

$$\left(\prod_{i\in A}(1-x_i) \prod_{i\in U\setminus A}(1+x_i)\right) \cdot g_\ell(x) =$$

$$\left(\sum_{I\subseteq U}(-1)^{|I\cap A|} y_I\right) \cdot \left(\sum_J g_\ell(J) y_J\right) = \sum_{I\subseteq U}(-1)^{|I\cap A|}\left(\sum_J g_\ell(J) y_{I\Delta J}\right)$$

$$= \sum_{I\subseteq U}(-1)^{|I\cap A|} g_\ell * y_I = (g_\ell * y)^T y^A$$

$$\geq 0 \quad \text{for all } A \subseteq U$$

This means: $M_U(g_\ell * y) \succeq 0$ for all $|U| = t$.

Analogously: $M_U(y) \succeq 0$ for all $|U| = t+1$

Hence $S_t(K) =$ projection of the above SDP conditions

$$\boxed{Q_{t+v}(K) \subseteq S_t(K)}$$

Algebraic background
Primal approach: Moment sequences

Problem: Minimize a polynomial $g(x) = \sum_\alpha g_\alpha \overbrace{x_1^{\alpha_1} \cdots x_n^{\alpha_n}}^{x^\alpha}$ over a semi-algebraic set:

$$K = \{x \in \Re^n \mid g_1(x) \geq 0, \ldots, g_m(x) \geq 0\}$$

$$\boxed{\begin{aligned} p^* &:= \min g(x) \text{ s.t. } x \in K \\ &= \min_{\substack{\mu \text{ probability} \\ \text{measure on } K}} \int_K g(x) d\mu(x) \end{aligned}}$$

Note:

$$\int_K g(x)d\mu(x) = \int_K (\sum_\alpha g_\alpha x^\alpha) d\mu(x) = \sum_\alpha g_\alpha \underbrace{\int_K x^\alpha d\mu(x)}_{y_\alpha} =: \sum_\alpha g_\alpha y_\alpha$$

$$\boxed{\begin{aligned} p^* = \; & \min \; \sum_\alpha g_\alpha y_\alpha \\ & \text{s.t. } y \text{ is the sequence of moments of} \\ & \quad\;\; \text{a probability measure on } K \end{aligned}}$$

Lower bound:

$$\boxed{\begin{aligned} p^* \geq p_t := \; & \min \sum_\alpha g_\alpha y_\alpha \\ & \text{s.t. } y_\emptyset = 1, \; M_t(y) \succeq 0, \; M_{t-v_\ell}(g_\ell * y) \succeq 0 \; \forall \ell \end{aligned}}$$

Here: $M_t(y) = (y_{\alpha+\beta})$ is indexed by integer sequences $\alpha \in Z_+^n$ with $\sum_i \alpha_i \leq t$

Dual approach: Sums of squares of polynomials

$$\begin{aligned} p^* &:= \min g(x) \text{ s.t. } x \in K \\ &= \max\ \lambda \text{ s.t. } g(x) - \lambda \text{ nonnegative on } K \end{aligned}$$

Lower bound:

$$\begin{aligned} p^* \geq \sigma_t := \;& \max\ \lambda \\ & \text{s.t. } g(x) - \lambda = p_0(x) + \sum_{\ell=1}^{m} p_\ell(x) g_\ell(x) \\ & \text{where } p_0, \ldots, p_m \text{ are sums of squares of} \\ & \text{polynomials with } \deg(p_0) \leq 2t, \\ & \text{and } \deg(p_\ell) \leq 2(t - v_\ell) \end{aligned}$$

Weak SDP duality: $\sigma_t \leq p_t \leq p^*$

Asymptotic convergence of σ_t to p^* as $t \to \infty$
[Lasserre 2000]

Theorem: [Putinar 1993] Every polynomial *positive* on K compact (+...) has a decomposition $p_0(x) + \Sigma_\ell\, p_\ell(x) g_\ell(x)$ where $p_0, \ldots, p_m$ are sums of squares of polynomials.

Finite convergence in n steps in the ± 1 case,
i.e., when the constraints $x_i^2 = 1$ are present in the
description of K

Algebraic interpretation of the Sherali-Adams construction [Lasserrre 02]

$p^* :=$ min. $g(x)$ over $K = \{x \mid g_1(x) \geq 0 \ldots g_m(x) \geq 0\}$

(1) Consider the products $g_1(x)^{\beta_1} \cdots g_m(x)^{\beta_m} \geq 0$ for all $\beta_1, \ldots, \beta_m \in Z_+$ with $\sum_i \beta_i \leq t$

(2) Linearize: $y_\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n} \quad \rightsquigarrow$ LP in variable y

Set: $\rho_t :=$ minimum of $\sum_\alpha g_\alpha y_\alpha$ over this LP

By LP duality:

$$\begin{aligned} \rho_t = \ & \max \ \lambda \\ & \text{s.t.} \quad g(x) - \lambda = \sum_{\beta \in Z_+^m,\ \sum_j \beta_j \leq t} \lambda_\beta g_1(x)^{\beta_1} \cdots g_m(x)^{\beta_m} \\ & \qquad \text{for some } \lambda_\beta \geq 0 \end{aligned}$$

Lower bound: $p^* \geq \rho_t$

Asymptotic convergence of ρ_t to p^* as $t \to \infty$ when K is a *polytope*

Theorem: [Handelman 1988] Every polynomial *positive* on a polytope K has a decomposition $\sum_\beta \lambda_\beta g_1^{\beta_1} \cdots g_m^{\beta_m}$ for some $\lambda_\beta \geq 0$

Several SDP relaxations for $\mathrm{CUT}(G)$

• Apply the Lovász-Schrijver construction to $K = \mathrm{MET}(G)$

$\rightsquigarrow N_+^t(G) \subseteq N_+^t(\mathrm{MET}(G))$

no explicit description ...

• Apply the Lasserre construction to $K = \mathrm{MET}(G)$

$\rightsquigarrow Q_t(\mathrm{MET}(G))$

too many constraints ...

• Apply the Lasserre construction to $K = [-1,1]^n$ and project on $\Re^E$

$\rightsquigarrow Q_t(G)$

the best choice!

Theorem: [La 01] $\boxed{Q_{t+2}(G) \subseteq N_+^t(G)}$

A more concise formulation for $Q_t(G) =$ projection on $\Re^E$ of $\{y \mid y_\emptyset = 1,\ M_t(y) \succeq 0\}$

$$M_t(y) = \begin{array}{c} \\ |I| \le t \\ \equiv t \\ \\ |I| \le t \\ \not\equiv t \end{array} \begin{array}{c} \begin{array}{cc} |I| \le t, \equiv t & |I| \le t, \not\equiv t \end{array} \\ \begin{pmatrix} A & B \\ B^T & C \end{pmatrix} \end{array}$$

- C is a principal submatrix of A
- may assume $B = 0$ as we project on pairs (even sets)

$\rightsquigarrow$ restrict to moment matrices indexed by sets $|I| \equiv t$ mod 2

$$M_1(y) = \begin{array}{c} \\ 1 \\ \vdots \\ n \end{array} \begin{array}{c} 1 \ldots n \\ \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix} \end{array} \quad M_2(y) = \begin{pmatrix} & & & \\ & & & \\ & & & \\ & & & \end{pmatrix}$$

$t = 1$: basic SDP relaxation: $Y \succeq 0$, diag(Y)=1

$t = 2$: SDP relaxation: $Y \succeq 0$, diag(Y)=1

$Y_{12,13} = Y_{\emptyset,23}$, $Y_{12,34} = Y_{13,24} = Y_{14,23}$

$\rightsquigarrow$ **triangle inequalities** are satisfied [AW01]

$$X = \begin{array}{c} \\ \emptyset \\ 12 \\ 13 \\ 23 \end{array} \begin{array}{c} \begin{array}{cccc} \emptyset & 12 & 13 & 23 \end{array} \\ \begin{pmatrix} 1 & y_{12} & y_{13} & y_{23} \\ y_{12} & 1 & y_{23} & y_{13} \\ y_{13} & y_{23} & 1 & y_{12} \\ y_{23} & y_{13} & y_{12} & 1 \end{pmatrix} \end{array} \succeq 0 \Longrightarrow e^T X e = 4(1+y_{12}+y_{13}+y_{23}) \ge 0$$

Properties of the SDP relaxation $Q_t(G)$

Definition: The *rank* $\rho(G)$ of graph G is the smallest integer t for which $\mathrm{CUT}(G) = Q_t(G)$.

$$\boxed{\rho(K_3) = \rho(K_4) = 2,\ \rho(K_5) = \rho(K_6) = 3,\ \rho(K_7) = 4}$$

Proposition: ρ is minor monotone

$$\rho(G) \le 1 \iff G \text{ has no } K_3\text{-minor}$$
$$\rho(G) \le 2 \iff G \text{ has no } K_5\text{-minor}$$
$$\rho(G) \le 3 \implies G \text{ has no } K_7\text{-minor}$$

Other minimal forbidden minors?

Proposition: $\rho(G/e) \le t \implies \rho(G) \le t + 1$

Conjecture: $\rho(K_n) = \lceil \frac{n}{2} \rceil$

Theorem: $\rho(K_n) \ge \lceil \frac{n}{2} \rceil$

Note: Enough to show the theorem for n odd and the conjecture for n even

Sketch of proof of $\rho(K_n) \geq \lceil \frac{n}{2} \rceil$ ($n = 2k+1$ odd)

Goal: show strict inclusion $\mathrm{CUT}(K_n) \subset Q_k(K_n)$

Show:

$$\min_{Q_k(K_n)} \sum_{ij} y_{ij} \stackrel{?}{=} -\frac{n}{2} < \min_{\mathrm{CUT}(K_n)} \sum_{ij} y_{ij} = \frac{1-n}{2}$$

For this: Construct $M_k(y) \succeq 0$ with $\sum_{ij} y_{ij} = -\frac{n}{2}$

$$a_0 := 1, \; a_{2r+2} := -a_{2r}\frac{2r+1}{n-2r-1}$$

$$y_I := a_{|I|} \quad \text{for all even sets } I$$

Theorem: $M_k(y) \succeq 0$

Proof:

(1) $Z :=$ principal submatrix of $M_k(y)$ indexed by the k-subsets of $\{1, \ldots, n-1\}$; $D :=$ order of Z.
Show that Z is positive definite.
Hence $M_k(y)$ has at least D positive eigenvalues.

Tools: Z belongs to the Johnson scheme $J(2k, k)$; compute its eigenvalues using hypergeometric series.

(2) Show that $M_k(y)$ has at least $N - D$ zero eigenvalues.

(1) Show that Z is positive definite

$$Z = \sum_{\ell=0}^{k} a_{2\ell} A_\ell$$

where A_ℓ are the 0/1 adjacency matrices of the Johnson scheme $J(2k,k)$, with (I,J) entry 1 iff $|I \Delta J| = 2\ell$

The **distinct eigenvalues** of Z are, for $u = 0, \ldots, k$,

$$\lambda_u := \sum_{\ell=0}^{k} a_{2\ell} \left(\sum_{j=0}^{\ell} (-1)^j \binom{u}{j} \binom{k-u}{\ell-j}^2 \right)$$

$$= \sum_{i=0}^{k} \binom{k-u}{i}^2 \left(\sum_{j=0}^{k} a_{2i+2j} (-1)^j \binom{u}{j} \right)$$

Show:

(i) The inner sum is equal to $a_{2i} \frac{(-k-1/2)_u}{(i-k)_u}$

(ii) $\lambda_u = \frac{(-k-1/2)_u}{(i-k)_u} \frac{(1/2)_u}{(k-u)!} = \frac{1 \cdot 3 \cdot \ldots \cdot (2k+1)}{2^k \cdot k! \cdot (2k-2u+1)} > 0$

Tool: *Gauss identity for hypergeometric series:* when $b \leq 0$ is integer and $x = 1$

$$\boxed{\sum_{i \geq 0} \frac{(a)_i (b)_i}{(c)_i} \frac{x^i}{i!} = \frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)}}$$

$$\Gamma(n+1) = n!, \quad \frac{\Gamma(x+n)}{\Gamma(x)} = (x)_n \text{ for integer } n \geq 0$$

See: $A = B$, Petkovsek-Wilf-Zeilberger 1996; Complexity of semi-algebraic proofs, Grigoriev-Hirsch-Pasechnik 2001

A curiosity about the spectrum of $M_k(y)$

The distinct eigenvalues of $M_k(y)$ are:

$n=3$ $k=1$	$n=5$ $k=2$	$n=7$ $k=3$	$n=9$ $k=4$	$n=11$ $k=5$
0	0	0	0	0
$\frac{3}{2}$	$\frac{5}{4}\cdot\frac{3}{2}$	$\frac{7}{6}\cdot\frac{5}{4}\cdot\frac{3}{2}$	$\frac{9}{8}\cdot\frac{7}{6}\cdot\frac{5}{4}\cdot\frac{3}{2}$	$\frac{11}{10}\cdot\frac{9}{8}\cdot\frac{7}{6}\cdot\frac{5}{4}\cdot\frac{3}{2}$
	$\frac{13}{8}$	$\frac{7}{6}\cdot\frac{13}{8}$	$\frac{9}{8}\cdot\frac{7}{6}\cdot\frac{13}{8}$	$\frac{11}{10}\cdot\frac{9}{8}\cdot\frac{7}{6}\cdot\frac{13}{8}$
			$\frac{263}{128}$	$\frac{11}{10}\cdot\frac{263}{128}$

The *new* eigenvalues $\frac{13}{8}$ and $\frac{263}{128}$ have multiplicity **one**

A tentative iterative proof for $Y = M_k(y) \succeq 0$

Say $n = 2k+1$ with k odd

$$Y = \begin{pmatrix} Y_{11} & Y_{13} & Y_{15} & \cdots \\ Y_{31} & Y_{33} & Y_{35} & \cdots \\ Y_{51} & Y_{53} & Y_{55} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$\boxed{Y_{11} = \tfrac{n}{n-1} E_1^{(1)}, \quad Y_{i1} E_0^{(1)} = 0 \ \text{ in } J(n,1)}$$

$$Y \succeq 0 \Longleftrightarrow$$

$$Y' := \begin{pmatrix} Y_{33} & Y_{35} & \cdots \\ Y_{53} & Y_{55} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} - \tfrac{n-1}{n} \begin{pmatrix} Y_{31} \\ Y_{51} \\ \vdots \end{pmatrix} \begin{pmatrix} Y_{31} \\ Y_{51} \\ \vdots \end{pmatrix}^T \succeq 0$$

$$\boxed{Y'_{33} = \tfrac{n(n-2)(n-4)}{(n-1)(n-3)(n-5)} E_3^{(3)}, \quad Y'_{i3} E_u^{(3)} = 0 \ (u \leq 2) \ \text{ in } J(n,3)}$$

$$Y' \succeq 0 \Longleftrightarrow$$

$$Y'' := \begin{pmatrix} Y'_{55} & \cdots \\ \vdots & \ddots \end{pmatrix} - \tfrac{(n-1)(n-3)(n-5)}{n(n-2)(n-4)} \begin{pmatrix} Y'_{53} \\ \vdots \end{pmatrix} \begin{pmatrix} Y'_{53} \\ \vdots \end{pmatrix}^T \succeq 0$$

$$\boxed{Y''_{55} = \tfrac{n(n-2)(n-4)(n-6)(n-8)}{(n-1)(n-3)(n-5)(n-7)(n-9)} E_5^{(5)} \ldots \ \text{ in } J(n,5)}$$

computations too hard ...

Geometric properties of moment matrices

$$\begin{array}{ll} \max & \frac{1}{2}\sum\limits_{ij} w_{ij}(1-y_{ij}) \\ \text{s.t.} & y_{\emptyset}=1,\ M_t(y)\succeq 0 \end{array}$$

When does this SDP relaxation solve Max-Cut exactly?

If $M_t(y) \succeq 0$, when is $M_1(y)$
a convex combination of cut matrices?

Yes if rank $M_1(y) = 1$

Yes if rank $M_1(y) \leq t$

Theorem: [La 01] If $M_t(y) \succeq 0$ and $M_1(y)$ has rank $\leq t$ then $M_t(y)$ is a convex combination of 2^{t-1} cut matrices.

$t = 1$ trivial
$t = 2$ [Anjos-Wolkowicz 2001]

Recall:

Conjecture: $\rho(K_n) \leq \lceil \frac{n}{2} \rceil$

Equivalently: If $M_{\lceil \frac{n}{2} \rceil}(y) \succeq 0$, then $M_1(y)$ is a convex combination of cuts

Sketch of proof

$Y = M_t(y) \succeq 0$, rank $M_1(y) \le t$, $1 \le t \le n-1$
$\Longrightarrow Y$ is convex combination of 2^{t-1} cut matrices

Proposition: If $t = n-1$ then $y_I = \pm 1$ for some even set $I \neq \emptyset$

Corollary: The theorem holds for $t = n-1$

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Prove the theorem by induction on $n \ge t+1$.
We can assume that $n \ge t+2$. Then $y_{I_0} = \pm 1$ for some even set $I_0 \neq \emptyset$ (with $|I_0| \le t+1$). Say, $n \in I_0$.

$$Y = \begin{pmatrix} \\ \\ \\ \end{pmatrix}$$

The induction assumption implies that

$$Y_0 = \sum_{A \in \mathcal{A}} \lambda_A M_t(y^A)$$

Goal: Show that each $A \in \mathcal{A}$ can be extended to $A' := A$ or $A \cup \{n\}$ in such a way that

$$Y = \sum_{A \in \mathcal{A}} \lambda_A M_t(y^{A'})$$

Tool: Use the structure of the set of even sets I for which $y_I = \pm 1$

Inequality	Min. over $\mathrm{CUT}(K_7)$	Min. over $Q_3(K_7)$	Min. over $Q_2(K_7)$	Min. over F_7	Min. over $Q_1(K_7)$	Min. over $N_+(K_7)$
triangle (1)	-1	**-1**	**-1**	**-1**	-1.5	**-1**
pentagonal (2)	-2	**-2**	-2.5	-2.5	-2.5	**-2**
hexagonal (3)	-4	**-4**	-4.5	-4.5	-4.5	-49/12 ~ -4.0833
(4)	-3	-3.5	-3.5	-3.5	-3.5	?
(5)	-6	-6.051882	-6.5	-6.5	-6.5	?
(6)	-7	**-7**	-7.5	-7.5	-7.5	?
bicycle (7)	-4	**-4**	-5	-5.0045	-5.8090	-4
(8)	-6	**-6**	-6.5817	-6.6522	-7.9661	?
(9)	-9	**-9**	-9.6433	-9.7036	-11.0166	?
parachute (10)	-4	**-4**	-4.7439	-4.8099	-5.9220	-4
grishukhin (11)	-5	**-5**	-5.6152	-5.7075	-6.9518	?

Comparing the strength of the various SDP relaxations for the facet defining inequalities of $\mathrm{CUT}(K_7)$

$$Q_2(K_7) \subseteq F_7 \subseteq \mathrm{MET}(K_7) \cap Q_1(K_7)$$

$$N_+(K_7) \subseteq F_7$$

Inequality	$\Sigma_{ij} c_{ij}$	ρ_3	ρ_2	ρ_F	ρ_1
triangle (1)	3	1	1	1	$\frac{8}{9} \sim$ **0.888**
pentagonal (2)	10	1	0.96	0.96	0.96
hexagonal (3)	20	1	0.979	0.979	0.979
(4)	21	**0.979**	0.979	0.979	0.979
(5)	34	0.998	0.987	0.987	0.987
(6)	33	1	0.987	0.987	0.987
bicycle (7)	16	1	**0.952**	**0.952**	0.917
(8)	30	1	0.984	0.982	0.948
(9)	47	1	0.988	0.987	0.965

Comparing the integrality ratios for the facets of K_7

Given $c \in \Re^{E_n}$ and $t \geq 0$

$$\rho_t := \frac{\Sigma_{ij} c_{ij} - \min(c^T y \mid y \in \mathrm{CUT}(K_n))}{\Sigma_{ij} c_{ij} - \min(c^T y \mid y \in Q_t(K_n))}$$

$$\rho_1 \geq 0.878 \quad \text{for } c \geq 0$$