Double Scaling Limit in Random Matrix Models and Semiclassical Asymptotics of Orthogonal Polynomials at Critical Points

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Workshop **"Mathematical Semi-Classical Analysis"**

MSRI, May 5-9, 2003

Introduction

We consider the unitary ensemble of random matrices,

$$d\mu_N(M) = Z_N^{-1} e^{-N \mathrm{Tr}\ V(M)} dM,$$

where

$$dM = \prod_{1\le i<j\le N} d\Re M_{ij} d\Im M_{ij} \prod_{i=1}^{N} dM_{ii}$$

and

$$Z_N = \int_{\mathcal{H}_N} e^{-N \mathrm{Tr}\ V(M)} dM,$$

on the space $\mathcal{H}_N$ of Hermitian $N \times N$ matrices $M = \left(M_{ij}\right)_{1\le i,j\le N}$. Here $V(x)$ is a polynomial,

$$V(x) = v_p x^p + v_{p-1} x^{p-1} + \dots,$$

of an even degree p with $v_p > 0$.

We will be interested in correlations between eigenvalues $\lambda_1 \leq \ldots \leq \lambda_N$ of M when N is large. The ensemble of eigenvalues is given by the Weyl formula,

$$d\mu_N(\lambda) = \widetilde{Z}_N^{-1} \prod_{i<j} |\lambda_i - \lambda_j|^2 \prod_{i=1}^{N} e^{-NV(\lambda_i)} d\lambda,$$

where

$$\widetilde{Z}_N = \int \prod_{i<j} |\lambda_i - \lambda_j|^2 \prod_{i=1}^{N} e^{-NV(\lambda_i)}.d\lambda,$$

It is convenient to rewrite it in the Gibbs form,

$$d\mu_N(\lambda) = \widetilde{Z}_N^{-1} e^{-NH_N(\lambda)} d\lambda,$$

where

$$H_N(\lambda) = -\frac{2}{N} \sum_{1 \leq j < k \leq N} \log|\lambda_j - \lambda_k| + \sum_{j=1}^{N} V(\lambda_j)\,.$$

Let $d\nu_N(x) = \rho_N(x)dx$ be the distribution of the eigenvalues on the line, so that for any $a < b$,

$$\int \left[\frac{1}{N}\#\{j : a < \lambda_j \leq b\}\right] d\mu_N(\lambda) = \int_a^b d\nu_N(x)\,.$$

As $N \to \infty$, there exists a weak limit of $d\nu_N(x)$,

$$d\nu_\infty(x) = \lim_{N\to\infty} d\nu_N(x)\,,$$

and the limiting distribution, $d\nu_\infty(x)$, minimizes the energy functional on the space of probability measures on the line,

$$I(d\nu(x)) = -\iint_{\mathbb{R}^2} \log|x-y| d\nu(x)\, d\nu(y) + \int_{\mathbb{R}} V(x) d\nu(y)\,.$$

Thus, $d\nu_\infty(x)$ is the *equilibrium measure* of the energy functional.

A rigorous proof of the existence of the limit $\lim_{N\to\infty} d\nu_N(x) = d\nu_{\text{eq}}(x)$, was given by A. Boutet de Monvel, L. Pastur, and M. Shcherbina, and by K. Johansson.

A. Boutet de Monvel, L. Pastur, and M. Shcherbina, On the statistical mechanics approach in the random matrix theory: Integrated density of states, J. Statist. Phys. **79** (1995), 585–611.

K. Johansson, On fluctuations of eigenvalues of random hermitian matrices, Duke Math. J. **91** (1998), 151–204.

For the existence and uniqueness of the equilibrium measure and its analytic properties see also

E. Saff and V. Totik, Logarithmic potentials and external fields, Springer-Verlag, New York, 1997.

P. Deift, T. Kriecherbauer, K. T-R. McLaughlin, New results on the equilibrium measure for logarithmic potentials in the presence of an external field, J. Appr. Theory **95** (1998) 399-475.

Properties of the equilibrium measure

- $d\nu_{\text{eq}}(x)$ is supported by a finite number of segments $[a_j, b_j]$, $j = 1, \dots, q$, and it is absolutely continuous with respect to the Lebesgue measure, $d\nu_{\text{eq}}(x) = \rho(x)dx$;

- the density function $\rho(x)$ is of the form

$$\rho(x) = \frac{1}{2\pi i} h(x) R_+^{1/2}(x),$$
$$R(x) = \prod_{j=1}^{q} (x - a_j)(x - b_j),$$

where $h(x)$ is a polynomial of the degree, $\deg h = p - q - 1$, and $R_+^{1/2}(x)$ means the value on the upper cut of the principal sheet of the function $R^{1/2}(z)$ with cuts on J,

$$J = \cup_{j=1}^{q} [a_j, b_j] \,.$$

The equilibrium measure is uniquely determined by the Euler-Lagrange conditions: for some real constant l,

$$2 \int_{\mathbb{R}} \log |x - s| \, d\nu_{\text{eq}}(s) - V(x) = l, \qquad \text{for} \quad x \in \cup_{j=1}^{q} [a_j, b_j] \, ;$$

$$2 \int_{\mathbb{R}} \log |x - s| \, d\nu_{\text{eq}}(s) - V(x) \leq l, \qquad \text{for} \quad x \in \mathbb{R} \setminus \cup_{j=1}^{q} [a_j, b_j] \, .$$

See

P. Deift, T. Kriecherbauer, K. T-R. McLaughlin, S. Venakides, and X. Zhou, Uniform asymptotics for polynomials orthogonal with respect to varying exponential weights and applications to universality questions in random matrix theory, *Commun. Pure Appl. Math.*, **52** (1999) 1335-1425

The Euler-Lagrange conditions imply that for $z \notin J$,

$$\omega(z) = \frac{V'(z)}{2} - \frac{h(z)R^{1/2}(z)}{2}, \qquad (*)$$

where

$$\omega(z) \equiv \int_J \frac{\rho(x)\,dx}{z-x} = z^{-1} + O(z^{-2}), \quad z \to \infty\,.$$

In addition, for any $j = 1, \ldots, q-1$,

$$\int_{b_j}^{a_{j+1}} \frac{h(x)R^{1/2}(x)}{2}\,dx = 0\,, \qquad (**)$$

which shows that $h(x)$ has at least one zero on each interval $b_j < x < a_{j+1}$; $j = 1, \ldots, q-1$.

From (*) we obtain that

$$\begin{aligned} V'(z) &= \mathrm{Pol}\left[h(z)R^{1/2}(z)\right], \\ \operatorname*{Res}_{z=\infty}\left[h(z)R^{1/2}(z)\right] &= -2, \end{aligned} \qquad (***)$$

and

$$h(z) = \mathrm{Pol}\left[\frac{V'(z)}{R^{1/2}(z)}\right],$$

where Pol $[f(z)]$ is the polynomial part of $f(z)$ at $z=\infty$. The latter equation expresses $h(z)$ in terms of $V(z)$ and the end-points, a_1, $b_1, \ldots,$ a_q, b_q.

The end-points can be further found from (***), which gives $q+1$ equation on $a_1, \ldots, b_q$ (observe that $\deg h = \deg V - q - 1$), and from (**), which gives the remaining $q-1$ equation.

The equilibrium measure $d\nu_{\rm eq}(x)$ is called *regular* (otherwise *singular*), see [DKMVZ], if

$$h(x) \neq 0 \quad \text{for} \quad x \in \cup_{j=1}^{q} [a_j, b_j]$$

and

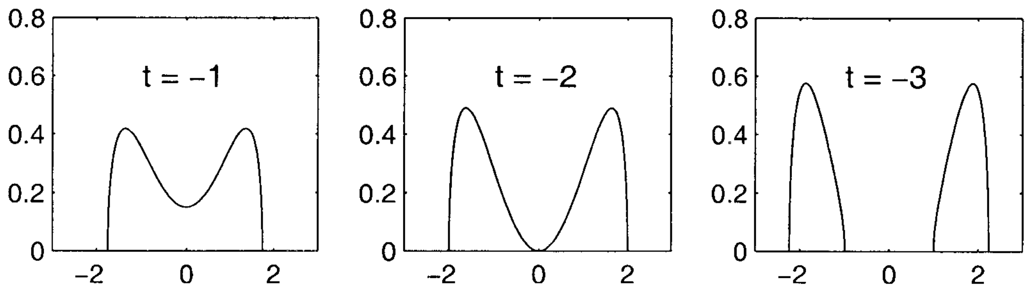
$$2 \int \log |x-s| \, d\nu_{\rm eq}(s) - V(x) < l, \qquad \text{for} \quad x \in \mathbb{R} \setminus \cup_{j=1}^{q} [a_j, b_j] \,.$$

The polynomial $V(x)$ is called *critical* if the corresponding equilibrium mesaure $d\nu_{\rm eq}(x)$ is singular. If $V(x)$ is a critical polynomial then the set S of its *singular points* consists of the points where either $h(x) = 0$, $x \in \cup_{j=1}^{q} [a_j, b_j]$, or

$$2 \int \log |x-s| \, d\nu_{\rm eq}(s) - V(x) = l, \qquad x \in \mathbb{R} \setminus \cup_{j=1}^{q} [a_j, b_j] \,.$$

Example. $V(x) = \frac{1}{4}x^4 + \frac{t}{2}x^2$.

Equilibrium measure density:



Here $t_c = -2$ is a critical point. In general, we will be interested in the following problems:

- Critical asymptotics of the free energy.

- Double scaling limit of correlation functions at the critical point.

- Double scaling limit of recurrence coefficients at the critical point.

Correlation functions. For any test function $\phi(x) \in C_0^\infty$, define

$$\phi[\lambda] = \sum_{j=1}^{N} \phi(\lambda_j)\,, \qquad \lambda = \{\lambda_1, \dots, \lambda_N\}\,.$$

The m-point correlation measure $d\nu_{mN}(x_1, \dots, x_m)$ is defined by the condition that for any m test functions $\phi_1(x), \dots, \phi_m(x)$,

$$\int_{\Lambda_N} \left(\prod_{j=1}^{m} \phi_j[\lambda] \right) d\mu_N(\lambda)$$
$$= \int_{\mathbb{R}^m} \left(\prod_{j=1}^{m} \phi_j(x_j) \right) d\nu_{mN}(x_1, \dots, x_m)\,.$$

In fact, $d\nu_N(x_1, \dots, x_m)$ is Lebesgue absolutely continuous on the set $\{x_j \neq x_k\}$,

$$d\nu_N(x_1, \dots, x_m) = K_{mN}(x_1, \dots, x_m)\, dx_1 \dots dx_m\,,$$

and $K_{mN}(x_1, \dots, x_m)$ is the m-point correlation function.

Scaling limits of correlation functions

Scaling limit in the bulk of the spectrum. Let $x \in \cup_{j=1}^{q}(a_j, b_j)$, so that the density of eigenvalues, $\rho(x) > 0$. The scaling limit of correlation functions is defined as

$$K_m(u_1, \ldots, u_m)$$
$$= \lim_{N\to\infty} \frac{1}{(N\rho(x))^m} K_{mN}\left(x + \frac{u_1}{N\rho(x)}, \ldots, x + \frac{u_m}{N\rho(x)}\right)$$

The problem is to prove the existence of the limit and to evaluate $K_m(u_1, \ldots, u_m)$.

Scaling limit at the regular end-point. Let x be a regular end-point, $x = a_j, b_j$, $h(x) \neq 0$. The scaling limit of correlation functions at the regular end-point is defined as

$$K_m(u_1, \ldots, u_m)$$
$$= \lim_{N\to\infty} \frac{1}{(cN^{2/3})^m} K_{mN}\left(x + \frac{u_1}{cN^{2/3}}, \ldots, x + \frac{u_m}{cN^{2/3}}\right)$$

The problem again is to prove the existence of the limit and to evaluate $K_m(u_1, \ldots, u_m)$ ($c > 0$ is an appropriate normalization factor).

Double scaling limit at the critical point. Let x be a critical point. The double scaling limit of correlation functions at the critical point is defined as

$$K_m(u_1,\dots,u_m;y) = \lim_{N\to\infty} \frac{1}{(cN^\kappa)^m}$$
$$\times K_{mN}\left(x+\frac{u_1}{cN^\kappa},\dots,x+\frac{u_m}{cN^\kappa};v_{cr}+\frac{y}{N^\gamma}\right),$$

where $\kappa,\gamma > 0$ are appropriate critical exponents, and $v_{cr}+\frac{y}{N^\gamma}$ is a perturbation of the coefficients of the critical polynomial $V_{cr}(x)$. The problem is to determine κ and γ, to prove the existence of the limit, and to evaluate $K_m(u_1,\dots,u_m;y)$ ($c > 0$ is an appropriate normalization factor).

Correlation functions in terms of orthogonal polynomials

Consider monic orthogonal polynomials $P_n(x) = x^n + \ldots$ on the line with respect to the weight $e^{-NV(x)}$, so that

$$\int_{-\infty}^{\infty} P_n(x)P_m(x)e^{-NV(x)}dx = 0, \quad n \neq m.$$

Let

$$h_n = \int_{-\infty}^{\infty} P_n(x)^2 e^{-NV(x)}dx.$$

Set

$$\psi_n(x) = \frac{1}{\sqrt{h_n}} P_n(x) e^{-\frac{NV(x)}{2}}.$$

Formula of Dyson-Mehta

$$K_{mN}(x_1, \ldots, x_m) = \det \left(Q_N(x_k, x_l)\right)_{k,l=1}^{m},$$

where $Q_N(x, y)$ is the Szegö kernel,

$$Q_N(x, y) = \sum_{n=0}^{N-1} \psi_n(x)\psi_n(y).$$

Scaling and double scaling limits for the Szegö kernel

The problem of the scaling limit in the bulk of the spectrum reduces to the one for the Szegö kernel,

$$
\begin{aligned}
&Q(u_1, u_2) \\
&= \lim_{N\to\infty} \frac{1}{N\rho(x)} Q_N\left(x + \frac{u_1}{N\rho(x)}, x + \frac{u_2}{N\rho(x)}\right),
\end{aligned}
$$

so that

$$K_m(x_1, \ldots, x_m) = \det\left(Q(x_k, x_l)\right)_{k,l=1}^{m},$$

Similarly, the scaling limit of the Szegö kernel at the regular end-point, $x = a_j,\ b_j$, is

$$Q(u_1, u_2) = \lim_{N\to\infty} \frac{1}{cN^{2/3}} Q_N\left(x + \frac{u_1}{cN^{2/3}}, x + \frac{u_2}{cN^{2/3}}\right),$$

and at the critical point,

$$Q(u_1, u_2; y) = \lim_{N\to\infty} \frac{1}{cN^{\kappa}} Q_N\left(x + \frac{u_1}{cN^{\kappa}}, x + \frac{u_2}{cN^{\kappa}}; v_{cr} + \frac{y}{N^{\gamma}}\right),$$

Main Results

Scaling limit in the bulk of the spectrum

Theorem **1.** *The scaling limit of the Szegö kernel in the bulk of the spectrum exists and is equal to the sine kernel,*

$$Q(u_1, u_2) = \frac{\sin \pi(u_1 - u_2)}{\pi(u_1 - u_2)}.$$

$V(x) = x^2$: Dyson, 1971.

$V(x) = \frac{1}{4}x^4 + \frac{t}{2}x^2$: Bleher, Its, 1997 (Riemann-Hilbert approach).

General $V(x)$: Boutet de Monvel, Pastur, Shcherbina 1998 (mean-field theory approach); Deift, Kriecherba McLaughlin, Venakides, and Zhou, 1999 (Riemann-Hilbert approach).

Scaling limit at the regular end-point

Theorem **2.** *The scaling limit of the Szegö kernel at a regular end-point exists and is equal to the Airy kernel,*

$$Q(u_1, u_2) = \frac{\mathrm{Ai}\,(u_1)\mathrm{Ai}\,'(u_2) - \mathrm{Ai}\,'(u_1)\mathrm{Ai}\,(u_2)}{(u_1 - u_2)}.$$

$V(x) = x^2$: Bowick and Brézin, 1991; Tracy, Widom, 1992.

$V(x) = \frac{1}{4}x^4 + \frac{t}{2}x^2$: Bleher, Its, 1997.

General $V(x)$: it follows from the results of Deift, Kriecherbauer, McLaughlin, Venakides, and Zhou, 1999.

Double scaling limit at a critical point

Theorem **3.** *Assume that $V(x) = \frac{1}{4}x^4 + \frac{t}{2}x^2$. The double scaling limit of the Szegö kernel at the critical point $t_c = -2$ exists and is equal to the Painlevé II kernel,*

$$\lim_{N\to\infty} \frac{1}{cN^{1/3}} Q_N\left(\frac{u_1}{cN^{1/3}}, \frac{u_2}{cN^{1/3}}; -2 + \frac{c_0 y}{N^{2/3}}\right) = Q(u_1, u_2; y),$$

where $c = 2^{-2/3}$, $c_0 = 2^{1/3}$,

$$Q(u_1, u_2; y) = \frac{\phi(u_1; y)\phi'(u_2; y) - \phi'(u_1; y)\phi(u_2; y)}{u_1 - u_2},$$

so that $\phi(u; y) = [\vec{\Phi}(u; y)]_1$, where $\vec{\Phi}(u; y)$ solves the 2×2 linear differential psi-equation associated with the critical solution to the Painlev'e II equation,

$$\vec{\Phi}'(u; y) = F(u; y)\vec{\Phi}(u; y),$$

$$\vec{\Phi}(u; y) \sim \begin{pmatrix} \cos\left(\frac{4u^3}{3} + yu\right) \\ -\sin\left(\frac{4u^3}{3} + yu\right) \end{pmatrix}, \quad |y| \to \infty.$$

The matrix $F(u; y)$ *is*

$$F(u; y) = \begin{pmatrix} 4q(y)u & 4u^2 + 2p(y) + r(y) \\ -4u^2 + 2p(y) - r(y) & 4p(y)u \end{pmatrix}.$$

where $q(y)$ *is the critical, Hastings-McLeod solution to the Painlevé II equation,*

$$q'' = yq + 2q^3,$$
$$q \sim \sqrt{-2y}, \quad y \to -\infty,$$
$$q \sim \mathrm{Ai}\ (y), \quad y \to \infty,$$

and

$$p(y) = q'(y), \quad r(y) = y + 2p^2(y).$$

Bleher and Its, 2002;

See also Baik, Deift, and Johansson, 1999 (circular ensemble, critical asymptotics of the recurrence coefficients).

Semiclassical Asymptotics for Orthogonal Polynomials

Christoffel-Darboux Formula

Three term recurrence relation:

$$x\psi_n(x) = \gamma_{n+1}\psi_{n+1}(x) + \beta_n\psi_n(x) + \gamma_n\psi_{n-1}(x).$$

If $Q : f(x) \to xf(x)$, then in the basis $\{\psi_n(x)\}$; Q is a symmetric tridiagonal matrix,

$$Q = \begin{pmatrix} \beta_0 & \gamma_1 & 0 & \dots \\ \gamma_1 & \beta_1 & \gamma_2 & \dots \\ 0 & \gamma_2 & \beta_3 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Christoffel-Darboux formula:

$$Q_N(x,y) = \sum_{n=0}^{N-1} \psi_n(x)\psi_n(y)$$
$$= \gamma_N \frac{\psi_N(x)\psi_{N-1}(y) - \psi_{N-1}(x)\psi_N(y)}{x-y}.$$

Lax Pair Equations

Define $\vec{\Psi}_n(x) = \begin{pmatrix} \psi_n(x) \\ \psi_{n-1}(x) \end{pmatrix}$.

Differential equation:

$$\vec{\Psi}'_n(x) = NA_n(x)\vec{\Psi}_n(x),$$

where

$$A_n(x) = \begin{pmatrix} -\frac{V'(x)}{2} - \gamma_n u_n(x) & \gamma_n v_n(x) \\ -\gamma_n v_{n-1}(x) & \frac{V'(x)}{2} + \gamma_n u_n(x) \end{pmatrix}$$

and

$$u_n(x) = [W(Q,x)]_{n,n-1},$$
$$v_n(x) = [W(Q,x)]_{nn},$$

where

$$W(Q,x) = \frac{V'(Q) - V'(x)}{Q - x}.$$

Recurrence equation:

$$\vec{\Psi}_{n+1}(x) = U_n(x)\vec{\Psi}_n(x),$$

where

$$U_n(x) = \begin{pmatrix} \gamma_{n+1}^{-1}(x - \beta_n) & -\gamma_{n+1}^{-1}\gamma_n \\ 1 & 0 \end{pmatrix}$$

Discrete String Equations

$$\begin{cases} [V'(Q)]_{nn} = 0, \\ \\ \gamma_n [V'(Q)]_{n,n-1} = \dfrac{n}{N} \,. \end{cases}$$

Lax pair for the quartic model

Let $V(x) = \frac{g}{4}x^4 + \frac{t}{2}x^2$, $g > 0$, and

$$\vec{\Psi}_n(z) \begin{pmatrix} \psi_n(z) \\ \psi_{n-1}(z) \end{pmatrix}.$$

Differential equation

$$\vec{\Psi}'_n(z) = NA_n(z)\vec{\Psi}_n(z),$$

where

$$A_n(z) = \begin{pmatrix} -\left(\frac{tz}{2} + \frac{gz^3}{2} + gzR_n\right) & R_n^{1/2}(gz^2 + \theta_n) \\ -R_n^{1/2}(gz^2 + \theta_{n-1}) & \frac{tz}{2} + \frac{gz^3}{2} + gzR_n \end{pmatrix}$$

and

$$\theta_n = t + gR_n + gR_{n+1}, \qquad R_n = \gamma_n^2.$$

Recurrence equation

$$zP_n(z) = P_{n+1}(z) + R_n P_{n-1}(z),$$

(orthogonal polynomials) or

$$\vec{\Psi}_{n+1} = U_n(z)\vec{\Psi}_n(z),$$

where

$$U_n(z) = \begin{pmatrix} R_{n+1}^{-1/2} z & -R_{n+1}^{-1/2} R_n^{1/2} \\ 1 & 0 \end{pmatrix}.$$

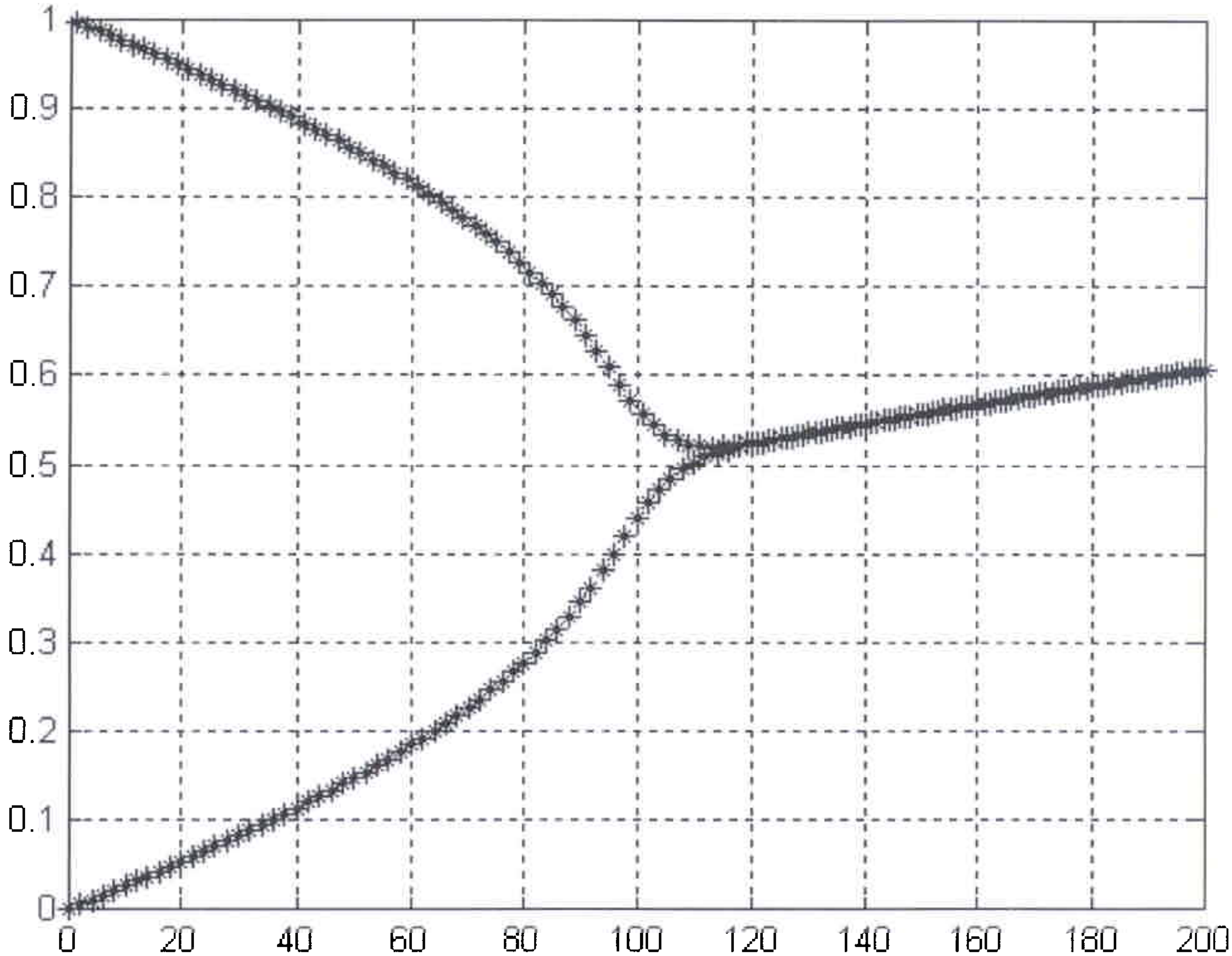
String equation

$$U_n' = N(A_{n+1}U_n - U_n A_n),$$

or

$$\frac{n}{N} = R_n(t + gR_{n-1} + gR_n + gR_{n+1}).$$

Recurrence coefficient R_n versus n ($g = 1$, $t = -1$, $N = 400$)



Scaling Limit Ansatz for the Recurrence Coefficients

$$R_n = a\left(\frac{n}{N}\right) + (-1)^n b\left(\frac{n}{N}\right), \qquad \frac{n}{N} < \lambda_c\,,$$

and

$$R_n = a\left(\frac{n}{N}\right), \qquad \frac{n}{N} > \lambda_c\,.$$

Critical point: $\lambda_c = \frac{t^2}{4g}$ $(t < 0)$.

Scaling functions (from the string equation):

$$a(\lambda) = -\frac{t}{2g}, \qquad b(\lambda) = \frac{\sqrt{t^2 - 4g\lambda}}{2g}, \qquad \lambda < \lambda_c,$$

and

$$a(\lambda) = \frac{-t + \sqrt{t^2 + 3g\lambda}}{6g}, \qquad \lambda > \lambda_c.$$

Double Scaling Limit Ansatz

$$R_n = -\frac{t}{2g} + N^{-1/3}(-1)^{n+1}c_1 q(y) \;\; +N^{-2/3}c_2 r(y) + O(N^{-1}),$$

where

$$y = c_0^{-1} N^{2/3}\left(\frac{n}{N} - \frac{t^2}{4g}\right),$$

$$c_0 = \left(\frac{t^2}{2g}\right)^{1/3}, \quad c_1 = \left(\frac{2|t|}{g^2}\right)^{1/3},$$

$$c_2 = \frac{1}{2}\left(\frac{1}{2|t|g}\right)^{1/3}.$$

Then the string equation reads

$$0 = R_n(t + gR_{n-1} + gR_n + gR_{n+1}) - \frac{n}{N}$$

$$= N^{-2/3}c_0\left(r - 2q^2 - y\right)$$

$$+N^{-1}(-1)^n\left(q'' - qr\right) + O\left(N^{-4/3}\right),$$

the Painlevé II on q, and $r = 2q^2 + y$.

Substitute the double scaling limit ansatz for R_n to the differential equation for orthogonal polynomials,

$$\vec{\Psi}'_n(z) = NA_n(z)\vec{\Psi}_n(z),$$

where

$$A_n(z) = \begin{pmatrix} -\left(\frac{tz}{2} + \frac{gz^3}{2} + gzR_n\right) & R_n^{1/2}(gz^2 + \theta_n) \\ -R_n^{1/2}(gz^2 + \theta_{n-1}) & \frac{tz}{2} + \frac{gz^3}{2} + gzR_n \end{pmatrix}$$

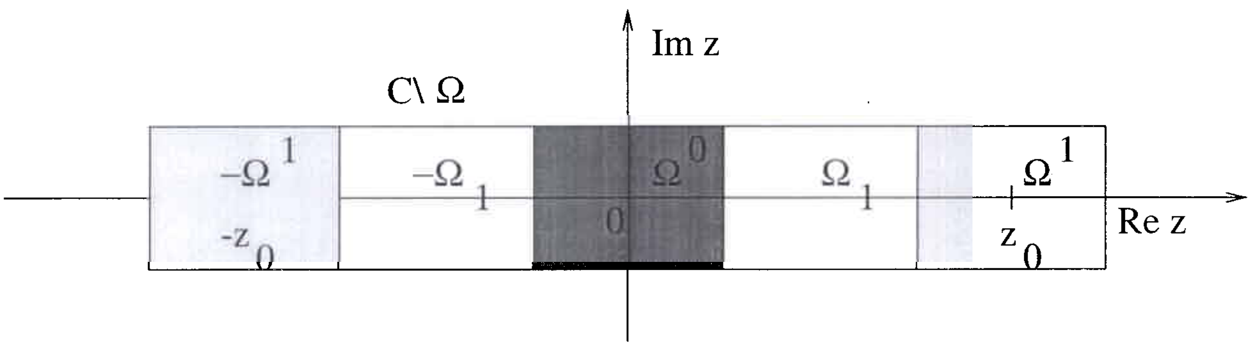
and

$$\theta_n = t + gR_n + gR_{n+1}, \qquad R_n = \gamma_n^2,$$

and solve it in the semiclassical approximation on the complex plane.

Semiclassical Solution on the Complex Plane

Divide the complex plane into regions:



Define $\Psi_n^0(z)$ as

$$\Psi_n^0(z) = \begin{cases} \Psi^{\mathsf{WKB}}(z), & z \in \mathbb{C} \setminus \Omega, \\ \Psi^{\mathsf{CP}}(z), & z \in \Omega^0, \\ \Psi^{\mathsf{TP}}(z), & z \in \Omega^1 \cup (-\Omega^1), \\ \Psi^{\mathsf{WKB}}(z), & z \in \Omega_1 \cup (-\Omega_1) \end{cases},$$

where $\Psi^{\mathrm{WKB}}(z)$ is the WKB solution, $\Psi^{\mathrm{TP}}(z)$ is the turning point solution (simple turning points), and $\Psi^{\mathrm{CP}}(z)$ is the critical point solution (4 merging turning points).

The estimate of the error term in both the double scaling Ansatz for recurrence coefficients and the semiclassical solution comes from the Riemann-Hilbert problem.

- **Conclusion**

Using the Riemann-Hilbert problem we proved the semiclassical asymptotics of orthogonal polynomials with respect to the exponential quartic weight in the double scaling limit. This proved the asymptotics for recurrence coefficients,

$$R_n = -\frac{t}{2g} + N^{-1/3}(-1)^n c_1 q(y) \quad + N^{-2/3} c_2 r(y) + O(N^{-1}),$$

where $q(y)$ is the Hastings-McLeod solution to Painlevé II,

$$\frac{n}{N} = \lambda_c + c_0 N^{-2/3} y,$$

and local asymptotics of correlations between eigenvalues in the double scaling limit:

- sine-kernel in the bulk of the spectrum,
- Airy-kernel at the edges of the spectrum,
- Painlevé II-kernel in the critical point.