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"Tight closure and completion"

$$A \longrightarrow \hat{A}$$

every ideal of A is tightly closed — not every ideal is tightly closed

A Gorenstein, non excellent.

A Gorenstein, F-regular $\Longleftrightarrow$ one parameter ideal is tightly closed.

I parameter ideal, $y \in A$ s/t

$\mathrm{soc}(A/I) = (y)$ $\qquad I = I^* \Longleftrightarrow y \notin I^*$

$$(I\hat{A})^* \neq I\hat{A} \Longleftrightarrow y \in \hat{I}^*$$

$$M_{\hat{I},y} = \{ c \in \hat{A} : \forall q = p^e > q_0 : cy^q \in \hat{I}^{[q]} \}$$

ideal of the multipliers

$$M_{\hat{I},y} \cap A = (0)$$

<u>Prop.</u> (A, m) loc. Noeth., $\mathrm{char}(A) = p > 0$

t a variable over A.

$I \subseteq A[\![t]\!]$ and $y \in A[\![t]\!] \Longrightarrow$

$M_{I,y} = A[\![t]\!]$ or $M_{I,y} \subseteq m\, A[\![t]\!]$

<u>Construction method</u> (W. Heinzer, S. Wiegand, —)

k a field, $\mathrm{char}(k) = p > 0$

$R = k[X, Y_1, \ldots, Y_n]_{(X, Y_1, \ldots, Y_n)}$

$\tau_1, \ldots, \tau_m \in X\, k[\![X]\!]$ alg. ind.

$\tau_i = \sum_{j=1}^{\infty} a_{ij} X^j$

$F(t_1, \ldots, t_m) \in R[t_1, \ldots, t_m]$

$\rho = F(\tau_1, \ldots, \tau_m) \in R[\tau_1, \ldots, \tau_m] \subseteq k[\![X, Y_1, \ldots, Y_n]\!]$

$\rho_n = \frac{1}{X^n}\left[\rho - F\left(\sum_{j=1}^{n} a_{1j} X^j, \ldots, \sum_{j=1}^{n} a_{mj} X^j\right)\right]$

$\in k[\![X, Y_1, \ldots, Y_n]\!]$

$\rho_n = X \rho_{n+1} + r_n \quad , \quad r_n \in R.$

$$B_n = R[\rho_n]_{(m_R,\,\rho_n)} \subseteq R[\rho_{n+1}]_{(m,\,\rho_{n+1})} \subseteq \cdots \hat{R}$$

$$B = \bigcup_{n \in \mathbb{N}} B_n \subseteq Q(R)(\rho)$$

$$\begin{array}{c}\cap\text{I} \\ A\end{array} = Q(R)(\rho) \cap \hat{R}$$

<u>Thm.</u> $\varphi : R[\rho]_{(m,\,\rho)} \longrightarrow \left(R[\tau_1,\ldots,\tau_m]_{(m,\,\tau_i)}\right)_x$

φ flat $\Longrightarrow$ $A = B$ and B is Noeth with completion $\hat{B} = \hat{R}$

k field of char $= p > 0$ s/t $p \equiv 1 \pmod 3$

<u>Model:</u> $R_0 = k[y, z, t]_{(y,z,t)}$

$D = R_0/(t^3 - z^3 - y^3)$ Gorenstein ring

$I = (y, z)$ soc $(D/I) = (t^2)$

$t^2 \in I^*$ $\therefore$ $p \equiv 1 \pmod 3$ $\therefore$ $M_{I,t^2} \subseteq m_D$

A s.t. $\hat{A} = \hat{D}[[x]]$

(I, x) parametric ideal

soc $\left(\hat{D}[[x]]/(I,x)\right) - (t^2)$

$M_{(I,x),t^2} \subseteq m_D \hat{D}[[x]]$.

k field of char $p > 0$ s/t $p \equiv 1 \pmod 3$

$$D = \frac{k[y,z,t]_{(y,z,t)}}{(t^3 - y^3 - z^3)}$$

$\tau_1, \tau_2, \tau_3 \in x k[[x]]$ alg. indep.

$$\tau_i = \sum_{n=1}^{\infty} u_{in} x^n$$

$$f - (t+\tau_1)^3 - (y+\tau_2)^3 - (z+\tau_3)^3$$

$\varphi : R[f] \longrightarrow R[\tau_1, \tau_2, \tau_3]$ flat.

So, $A = B/(\rho)$,

A is F-reg., but $\hat{A}$ is not

$\hat{A} \cong \hat{D}[\![X]\!]$ not F-reg.

$M_{(Z,X),t^2} \subseteq m_D \hat{D}[\![X]\!]$

$$\hat{A} = k[\![X,Y,Z,t]\!] \Big/ \underbrace{\left((t+\tau_1)^3 - (Y+\tau_2)^3 - (Z+\tau_3)^3\right)}_{=\rho}$$

$$= k[\![t+\tau_1, Y+\tau_2, Z+\tau_3, X]\!] \Big/ (\rho)$$

$$= k[\![t+\tau_1, Y+\tau_2, Z+\tau_3]\!] \Big/ (\rho) [\![X]\!].$$

By thm. $M_{(Y+\tau_2, Z+\tau_3, X), t^2} \subseteq (t+\tau_1, Y+\tau_2, Z+\tau_3)\hat{A}$

so, A is F-reg.

Using Heitmann's construction:

<u>Thm.</u> (T, m) complete, loc. Gorenstein ring of char $p > 0$. Suppose T is not F-reg. and $\hat{I} \subseteq T$ parameter ideal s/t

(*) $(y) = \mathrm{Soc}(T/\hat{I})$ and $y^q \notin I^{[q]}$ $\forall q = p^e$

$\Rightarrow$ $T[\![X]\!]$ is completion of an F-reg ring.