Ruchira Datta MSRI Comm. Alg. Workshop Notes

Asymptotic Constructions

Def Graded System of Ideals

X: a smooth affine variety over $\mathbb{C}$
A graded family of ideals $\mathfrak{a}_\bullet = \{\mathfrak{a}_k\}_{k \in \mathbb{N}}$
is a family of ideals $\mathfrak{a}_k \subseteq \mathbb{C}[X]$
s.t. $\mathfrak{a}_\ell \mathfrak{a}_m \subseteq \mathfrak{a}_{\ell+m}$, $\ell, m \geq 1$.
(We will also assume $\mathfrak{a}_k \neq (0)$ for $k \gg 0$.)

Example 0 Fix $\mathfrak{b} \subseteq \mathbb{C}[X]$, set $\mathfrak{a}_k = \mathfrak{b}^k$
(trivial example)

NB: Def implies "$\mathrm{Rees}(\mathfrak{a}_\bullet)$"
$\mathbb{C}[X]' \oplus \mathfrak{a}_1 \oplus \mathfrak{a}_2 \oplus \cdots$
is a graded algebra.
In interesting cases, not f.g. (finitely generated)

Example 1 V = projective variety
D = big divisor on V
$\mathfrak{b}_k = \mathrm{bs}(|kD|)$

Example 2 $Z \subseteq X$ a reduced subvariety, defined by a radical ideal $q \subseteq \mathbb{C}[X]$
Symbolic powers
$q^{(k)} = \{ f \in \mathbb{C}[X] \mid \mathrm{ord}_x f \geq k, \text{ gen } x \in Z \}$
form graded system $q^{\bullet} = \{q^{(k)}\}$

Example 3 Let v be $\mathbb{R}$-valued valuation on $\mathbb{C}(X)$ centered on $\mathbb{C}[X]$
$\mathfrak{a}_k = \{ f \in \mathbb{C}[X] \mid v(f) \geq k \}$

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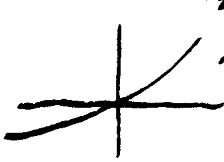
Example 3A: Take birational map
$\eta: Y \to X$ birational, Y smooth
$\cup\!|$
E prime divisor, $v(f) = \mathrm{ord}_E(f)$
$\mathfrak{a}_k = \mu_* \mathcal{O}_Y(-kE)$

Example 3B: In $\mathbb{C}[s,t]$ put $v(s)=1$, $v(t)=1/\sqrt{2}$
get valuation by "weighted degree"
$\mathfrak{a}_k = \left\{ \begin{array}{l} \text{monomial ideal generated by} \\ \text{all } s^i t^j \text{ s.t. } i + j/\sqrt{2} \geq k \end{array} \right\}$ $\leadsto$ "$(s, t^{\sqrt{2}})$" $\overset{\mathfrak{a}_\bullet}{\shortparallel}$

Example 3C: Given $f \in \mathbb{C}[s,t]$
$v(f) = \mathrm{ord}_z f(z, e^z - 1)$



$\mathfrak{a}_k = (s^k, t - P_{k-1}(s))$
where $P_k(t)$ = kth Taylor polyn of $e^z - 1$

Example: $\mathfrak{b} \subseteq \mathbb{C}[t_1, \ldots, t_d]$ $\quad \mathfrak{a}_k = \mathrm{in}(\mathfrak{b}^k)$
initial ideal

<u>Prop/Def</u> Given $\mathfrak{a}_\bullet = \{\mathfrak{a}_k\}$, fix an index ℓ. Then for $p \gg 0$, the multiplier ideals $\mathcal{J}(\mathfrak{a}_{\ell p}^{1/p})$ all coincide. This common ideal is the asymptotic multiplier ideal of $\mathfrak{a}_\bullet$ at level ℓ.
Write it as $\mathcal{J}(\|\mathfrak{a}_\ell\|)$ or $\mathcal{J}(\mathfrak{a}_\bullet^\ell)$.

<u>Idea</u>: Check that for $p, q > 0$, $\mathcal{J}(\mathfrak{a}_{\ell p}^{1/p}) \subseteq \mathcal{J}(\mathfrak{a}_{\ell pq}^{1/pq})$
By Noetherian condition, $\{\mathcal{J}(\mathfrak{a}_{\ell p}^{1/p})\}_{p>0}$ contains a unique maximal element. This is the stable ideal.

<u>Example 3B'</u> $\mathfrak{a}_\bullet$ = "$(s, t^{\sqrt{2}})$"
$\mathfrak{a}_k$ = (monomial ideal gen by $s^i t^j$ s.t. $i + j/\sqrt{2} \geq k$)

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$\mathcal{J}(\|\mathfrak{a}_\ell\|) = \langle s^i t^j \mid (i+1) + (j+1)/\sqrt{2} > \ell \rangle$

<u>Example 3C'</u> $v(f) = \mathrm{ord}_z f(z, e^z - 1)$
$\mathcal{J}(\mathfrak{a}_\bullet^\ell) = \mathbb{C}[s,t]$ for all ℓ (HW)
(Hint: Each $\mathfrak{a}_k$ contains a smooth curve.)

<u>Theorem</u> (Demailly, Ein, Lazarsfeld, Karen Smith)
Let $\mathfrak{a} = \{\mathfrak{a}_k\}$ be a graded family, and fix an index ℓ. Then for all m:
$\mathfrak{a}_\ell^m \subseteq \mathfrak{a}_{\ell m} \subseteq \mathcal{J}(\|\mathfrak{a}_{\ell m}\|) \underset{(!)}{\subseteq} \mathcal{J}(\|\mathfrak{a}_\ell\|)^m$

Specifically, if $\mathcal{J}(\|\mathfrak{a}_\ell\|) \subseteq \mathfrak{b}$ for some $\ell, \mathfrak{b}$, then $\mathfrak{a}_{\ell m} \subseteq \mathfrak{b}^m$ $\forall m$.

<u>Application to Symbolic Powers</u>

Consider reduced $Z \subseteq X$, $\mathfrak{q} \subseteq \mathbb{C}[X]$.

<u>Lemma</u> Assume all irreducible components of Z have codim $\leq \ell$. Then
(*) $\mathcal{J}(\|\mathfrak{q}^{(\ell)}\|) \subseteq \mathfrak{q}$

<u>Corollary</u> $\forall m > 0$, $\mathfrak{q}^{(m\ell)} \subseteq \mathfrak{q}^m$
In particular, if $d = \dim X$, $\mathfrak{q}^{(dm)} \subseteq \mathfrak{q}^m$ $\forall m$, $\forall$ radical $\mathfrak{q}$.

History: Irena Swanson showed $\exists k = k(\mathfrak{q})$ s.t. $\mathfrak{q}^{(km)} \subseteq \mathfrak{q}^m$. Hochster-Huneke proved for all local rings containing a field.

<u>Proof</u> Assume Z irreducible, having pure codim ℓ.
$\mathfrak{q}$ radical $\Rightarrow$ can test membership at generic point of Z.

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So can assume Z smooth. Then $q^{\langle k \rangle} = q^k$
$\mathcal{J}(\|q^{\langle \ell \rangle}\|) = \mathcal{J}(q^\ell) = q$ (compute on $Bl_q(X)$).
(localize at q, then compute there)

Proof of inclusion $\mathcal{J}(\|\mathfrak{a}_{m\ell}\|) \subseteq \mathcal{J}(\|\mathfrak{a}_\ell\|)^m$

(1°) "subadditivity" (Demailly, Ein, Lazarsfeld)
Fix $\mathfrak{a}, \mathfrak{b} \subseteq \mathbb{C}[X]$, $c, f > 0$. Then
$$\mathcal{J}(\mathfrak{a}^c \mathfrak{b}^f) \subseteq \mathcal{J}(\mathfrak{a}^c)\,\mathcal{J}(\mathfrak{b}^f)$$
So for $m \in \mathbb{N}$ $\quad \mathcal{J}(\mathfrak{a}^{cm}) \subseteq \mathcal{J}(\mathfrak{a}^c)^m$

(2°) Given $\mathfrak{a}_\bullet = \{\mathfrak{a}_k\}$, fix $p \gg 0$.
$$\mathcal{J}(\|\mathfrak{a}_{\ell m}\|) = \mathcal{J}(\mathfrak{a}_{\ell m p}^{1/p}) = \mathcal{J}(\mathfrak{a}_{\ell m p}^{m/pm})$$
$$\overset{(1°)}{\subseteq} \mathcal{J}(\mathfrak{a}_{\ell p m}^{1/pm})^m = \mathcal{J}(\|\mathfrak{a}_\ell\|)^m$$

Inputs to Proof of (1°)

(a) Restriction Thm (Esnault-Viehweg)

$X \supseteq Y$ a smooth subvariety
$\mathfrak{a} \subseteq \mathbb{C}[X]$, $Y \not\subseteq \mathrm{Zeroes}(\mathfrak{a})$

Theorem $\mathcal{J}(Y, (\mathfrak{a}\,\mathbb{C}[Y])^c) \subseteq \mathcal{J}(X, \mathfrak{a}^c)\,\mathbb{C}[Y]$

(b) Given $\mathfrak{a}, \mathfrak{b} \subseteq \mathbb{C}[X]$, take $c = d = 1$
Pass to $X \times X$. $\mathfrak{a} \odot \mathfrak{b} \overset{\text{def}}{=} pr_1^{-1}(\mathfrak{a})\, pr_2^{-1}(\mathfrak{b})$
$\mathcal{J}(X \times X, \mathfrak{a} \odot \mathfrak{b}) = pr_1^{-1}\mathcal{J}(\mathfrak{a})\, pr_2^{-1}\mathcal{J}(\mathfrak{b})$
Restrict to diagonal Δ.