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From numbers to q-numbers to elliptic numbers: The elliptic gamma function

1. Inveni terminum generalem
2. Difference equations with elliptic coefficients
3. Duality
4. Modular properties
5. Elliptic zeta values

1729 Euler to Goldbach: "I found the general term of the sequence $1, 2, 6, 24, 120, \cdots$:

$$\frac{1\cdot 2^m}{1+m}\cdot\frac{2^{1-m}3^m}{2+m}\cdot\frac{3^{1-m}4^m}{3+m}\;\frac{4^{1-m}5^m}{4+m}\cdot \text{etc} = \Gamma(m+1)$$

If $m=\frac{1}{2}$, I got $\frac{1}{2}\sqrt{\sqrt{-1}\,\ln(-1)}$ " $=\frac{1}{2}\sqrt{\pi}$ Euler

- q-numbers $\quad [n]_\sigma = \dfrac{\sin(n\sigma)}{\sin\sigma} \xrightarrow[\sigma\to 0]{} n.$

$$[n]_\sigma! = [1]_\sigma [2]_\sigma \cdots [n]_\sigma$$

- elliptic numbers $\quad [n]_{\sigma,\tau} = \dfrac{\theta_1(\sigma n,\tau)}{\theta_1(\sigma,\tau)} \xrightarrow[\tau\to i\infty]{} [n]_\sigma$

or θ_0.

$$\Gamma(z+1) = z\,\Gamma(z).$$

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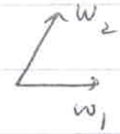
2. Difference eqns

$$u(z+\sigma) = a(z)\,u(z) \qquad a(z) \text{ is an elliptic function } \not\equiv 0.$$

$$a(z+w_i) = a(z) \qquad i = 1, 2$$



u_1, u_2 are solutions $\not\equiv 0$

$$\Rightarrow \quad u_1(z) = \lambda(z)\,u_2(z)$$

where $\lambda(z+\sigma) = \lambda(z)$.

Space of soln is 1-d v. space over field of periodic fns.

- First assume $a(z)$ rational

$$a(z) = a \cdot \frac{\prod (z - z_i)}{\prod (z - w_i)} \qquad a, z_i, w_i \in \mathbb{C}.$$

Solution of $u(z+\sigma) = a(z)\,u(z)$ is

$$u(z) = a^{z/\sigma}\, \frac{\prod \sigma\Gamma\left(\frac{z - z_i}{\sigma}\right)}{\prod \sigma\Gamma\left(\frac{z - w_i}{\sigma}\right)}$$

- $a(z)$ elliptic

$$a(z) = a \cdot \prod_{i=0}^{n} \frac{\theta_0\left(\frac{z-z_i}{\omega_1}, \frac{\omega_2}{\omega_1}\right)}{\theta_0\left(\frac{z-w_i}{\omega_1}, \frac{\omega_2}{\omega_1}\right)}$$

where $\theta_0(z,\tau) = \prod_{j=0}^{\infty} \left(1-q^{j+1}e^{-2\pi i z}\right)\left(1-q^{j}e^{2\pi i z}\right)$,

$$q = e^{2\pi i \tau}$$

<u>Joint work with A. Varchenko</u>

<u>Theorem</u> The equation $u(z+\sigma) = \theta_0(z,\tau)\,u(z)$ has a unique meromorphic solution so that

assume $\operatorname{Im}\sigma, \operatorname{Im}\tau > 0$

(i) $u(z+1) = u(z)$

(ii) $u(z)$ is holomorphic in $\operatorname{Im} z > 0$

(iii) $u\left(\frac{\tau+\sigma}{2}\right) = 1$

Formula

$$u(z) = \Gamma(z,\tau,\sigma) = \prod_{j=0}^{\infty}\prod_{k=0}^{\infty} \frac{\left(1-q^{j+1}r^{k+1}e^{-2\pi i z}\right)}{\left(1-q^{j}r^{k}e^{2\pi i z}\right)}$$

$$q = e^{2\pi i \tau}, \quad r = e^{2\pi i \sigma}, \quad \tau = \frac{\omega_2}{\omega_1}$$

(elliptic gamma function)

3. $u(z) \Rightarrow u(z+\omega_1)$, $u(z+\omega_2)$ are also solns.
So proportional to $u(z)$. (by periodic function)

$$\Gamma(z+1,\tau,\sigma) = \Gamma(z,\tau,\sigma)$$
$$\Gamma(z+\tau,\tau,\sigma) = \theta_0(z,\sigma)\Gamma(z,\tau,\sigma) \quad \text{"monodromy"}$$
$$\Gamma(z+\sigma,\tau,\sigma) = \theta_0(z,\tau)\Gamma(z,\tau,\sigma) \quad \text{"equation"}$$

$1, \tau, \sigma. \rightarrow$ some duality between periods τ, σ

4. $u(z+\sigma) = a(z)u(z)$ $\qquad a(z+\gamma) = a(z)$

$$\gamma \in L = \omega_1 \mathbb{Z} + \omega_2 \mathbb{Z}$$

Solution $u_b = \prod \frac{\Gamma}{\Gamma}$ depending on the choice of a basis $b = \{\omega_1, \omega_2\}$ of L

$$\operatorname{Im} \frac{\omega_2}{\omega_1} > 0$$

$$u_b(z) = u_{b'}(z) \underbrace{\lambda_{bb'}(z)}_{\sigma\text{-periodic}}$$

Need transformation properties under

$$(z,\tau,\sigma) \longmapsto \left(\frac{z}{c\tau+d}, \frac{a\tau+b}{c\tau+d}, \frac{\sigma}{c\tau+d}\right)$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$$

gen: $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

3-term relations

$$\boxed{\begin{aligned}&(\mathrm{Im}\tfrac{\sigma}{\tau}>0)\quad \Gamma\left(\tfrac{z}{\tau},-\tfrac{1}{\tau},\tfrac{\sigma}{\tau}\right)=e^{i\pi Q(z,\tau,\sigma)}\,\Gamma\left(\tfrac{z-\tau}{\sigma},-\tfrac{\tau}{\sigma},-\tfrac{1}{\sigma}\right)\Gamma(z,\tau,\sigma)\\ &\Gamma(z,\tau,\tau+\sigma)\,\Gamma(z+\tau,\tau,\sigma+\tau)=\Gamma(z,\tau,\sigma)\end{aligned}}$$

$Q(z,\tau,\sigma)\in\mathbb{Q}(\tau,\sigma)[z]$ of degree 3 in z.

Compare with

$$\theta_0\left(\tfrac{z}{\tau},-\tfrac{1}{\tau}\right)=e^{\pi i P(z,\tau)}\,\theta_0(z,\tau)$$

$$\theta_0(z,\tau+1)=\theta_0(z,\tau)$$

1-cocycle in $H^1(G_2, e^{2\pi i\mathbb{Q}(\tau)[z]})$

$$H^1(G_2, e^{2\pi i\mathbb{Q}(\tau)[z]})\longrightarrow H^2(G_2, 2\pi i\mathbb{Z}).$$

get nontrivial class.

$G_n = SL(n,\mathbb{Z})\ltimes\mathbb{Z}^n$

For Γ-functions,

Cocycle in $H^2(G_3, e^{2\pi i\mathbb{Q}(\tau,\sigma)[z]})$

$$\downarrow$$

$H^3(G_3, 2\pi i\mathbb{Z})$. nontrivial image.

5. $$\sigma_k(n) = \sum_{d|n} d^k \qquad \left(\text{e.g. } \sigma_k(6) = 1 + 2^k + 3^k + 6^k\right)$$

$$D_k(q) = \frac{(-2\pi i)^{k+1}}{k!} \sum_{n=1}^{\infty} \sigma_k(n)\, q^n$$

$$\lim_{q\to 1} \tau^{k+1} D_k(q) = \zeta(k+1) = \sum_{n=1}^{\infty} \frac{1}{n^{k+1}}$$

$$q = e^{2\pi i \tau}$$

If k is odd,

$$G_k(\tau) = 2\zeta(2k) + 2D_k(q)$$

$$G_k\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{2k} G_k(\tau)$$

What about even k?

Recall $$\ln \Gamma_{\text{Euler}}(z+1) = -\gamma z + \sum_{j=2}^{\infty} \frac{\zeta(j)}{j} (-z)^j$$

elliptic analog:

$$\ln \frac{\Gamma(z+\sigma, \tau, \sigma)}{\Gamma(\sigma, \tau, \sigma)} = \sum_{j=1}^{\infty} \frac{Z_j(\tau,\sigma)}{j} (-z)^j$$

$j \geq 4$ $$Z_j(\tau,\sigma) = Z_j(\tau, \tau+\sigma) + Z_j(\tau+\sigma, \sigma)$$

$$Z_j(\tau,\sigma) = \tau^{-j} Z_j\left(-\tfrac{1}{\tau}, \tfrac{\sigma}{\tau}\right) + \sigma^{-j} Z_j\left(\tfrac{\tau}{\sigma}, -\tfrac{1}{\sigma}\right)$$

$$Z_j(\tau,\sigma) = \left\{ \begin{array}{ll} D_{j-1}(q) - D_{j-1}(r) & j \text{ even} \\ D_{j-1}(q) + D_{j-1}(r) + 2 \sum\limits_{\gcd(a,b)=1} D_{j-1}(q^a r^b) & \end{array} \right\}$$

$$\xrightarrow[q \to 0]{} D_{j-1}(r).$$

Exercise (Euler) (in the same letter to Goldbach)

$$1, \quad 1+\frac{1}{2}, \quad 1+\frac{1}{2}+\frac{1}{3}, \quad 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}, \quad \cdots$$

$m=1$, $m=2$, ...

$m = -\frac{1}{2} \Rightarrow$ get $-2\ln 2$

$\ln 2 = 0.69314718056$