Chiral De Rham Complex

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string

$\{$  $\longrightarrow M \} = LM.$

$$\mathcal{H}_{Bos} = \mathcal{F}(LM)$$

$\Sigma =$ 

$$\downarrow$$
$$M$$

$$\mathcal{H}_{Fer} = \Omega(LM)$$

<u>Vertex algebra</u>

V – $\mathbb{C}$ – superspace

$$Y : V \longrightarrow (\mathrm{End}\, V)[[z, z^{-1}]]$$

$$Y(v, z) =: v(z)$$

$$v(z) = \sum v_{(n)} z^{-n-1}$$

$$\partial \in (\mathrm{End}\, V)_{\bar{0}}$$

$$|0\rangle \in V.$$

<u>axioms</u> $(\partial a)(z) = a(z)'$, $a \in V$

$$|0\rangle(z) = \mathrm{id}_V$$

locality $\displaystyle a(z)\, b(w) \underset{|z|>|w|}{=} \sum_{k=0}^{N} \frac{c_k(w)}{(z-w)^{k+1}} + \text{reg}.$

V - conformal if $\exists L \in V$, $L(z)L(w) = \frac{c}{2(z-w)^4} + \frac{2L(w)}{(z-w)^2}$

and $\partial = L_{-1}$, L_0 is diagonal on V with "integral" eigenvalues $+ \frac{\partial_w L}{z-w} + \text{reg.}$

$L_0(v) = hv$ h: conformal weight

New grading : $v(z) = \sum v_n z^{-n-h}$ $v_n = v_{(n+h-1)}$

$\Rightarrow$ $L(z) = \sum L_n z^{-n-2}$

$[L_m, L_n] = (m-n) L_{n+m} + \delta_{0,m+n} \frac{m^2-m}{12} c$

V is called $N=2$ (topological) vertex algebra of dim d if V is conformal with $c=0$

and $\exists J, G, Q$

$h=1$ even, $h=2$ odd, $h=1$ odd

s.t. $L \cdot J \sim \frac{-d}{(z-w)^3} + \frac{J}{(z-w)^2} + \frac{J'}{z-w}$

$L \cdot G \sim \frac{2G}{(z-w)^2} + \frac{G'}{z-w}$

$L \cdot Q \sim \frac{Q}{(z-w)^2} + \frac{Q'}{z-w}$

$J \cdot J \sim \frac{d}{(z-w)^2}$, $G \cdot G \sim 0$, $Q \cdot Q \sim 0$

There are other relations for $J \cdot G$, $J \cdot Q$, $Q \cdot G$.

There is another conformal structure

$$\tilde{L} = L - \partial J$$

New $N=2$ $\quad \tilde{Q} = G$

$$\tilde{G} = -Q$$

Ex Hisenberg L. algebra

$$H = \langle a_n, b_n, c \rangle_{n \in \mathbb{Z}}$$

$$a(z) = \sum a_n z^{-n-1} \quad , \quad b(z) = \sum b_n z^{-n}$$

$$a(z) \cdot b(w) \sim \frac{1}{z-w} \iff [a_m, b_n] = \delta_{m,-n} \, c$$

Vertex algebra. $\quad V =$ Fock repn of H.

$|0\rangle$ annihilated by $W = \langle a_{\geq 0}, b_{>0} \rangle$

So $\quad V = \mathbb{C}[a_{<0}, b_{\leq 0}] \, |0\rangle$ as v. space

$$b_0(z) = b(z) \qquad \underline{b}_{-n}(z) := \partial_z^{(n)} b(z)$$

$$a_{-1}(z) := a(z) \qquad \partial^{(n)} = \partial^n / n!$$

$$a_{-n-1}(z) = \partial_z^{(n)} a(z).$$

In Higher order monomials, use normal ordering

$$:xy: = \begin{cases} yx & \text{if } x \text{ is annihil} \\ xy & \text{otherwise.} \end{cases}$$

$x \in \{b_n, a_n\}_{n \in \mathbb{Z}}$

$y \in \operatorname{End} V$

now $\quad x_1 \cdots x_p |0\rangle \in V$

$$\downarrow$$

$$(x_1 \cdots x_p)(z) := \,:x_1(z) \cdots x_p(z):$$

$L = b_{-1} a_{-1}$ — Virasoro vector

$\hookrightarrow L(z) = \,:\partial_z b(z) \cdot a(z): \qquad c = 2$

Clifford algebra $\quad Cl = \langle \varphi_m, \psi_m \rangle_{m \in \mathbb{Z}}$

Fermionic Fock module Λ

$b \leftrightarrow \varphi$
$a \leftrightarrow \psi$

(all formulas stay the same mod $(-1)^{\text{some power}}$ signs.)

$$\varphi(z) = \sum \varphi_n z^{-n} \qquad \psi(z) = \sum \psi_n z^{-n-1}$$

$$\varphi(z) \cdot \psi(w) \sim \frac{1}{z-w} \qquad c = -2$$

Conf str. $\quad L = \varphi_{-1} \psi_{-1}$

Chiral de Rham cpx of $\mathbb{A}^1$

$\Omega := V \otimes \Lambda$ — vertex algebra

$$L = b_{-1} a_{-1} + \varphi_{-1} \psi_{-1}$$

$c = 0$. it has topological structure:

$J := \varphi_0 \psi_{-1}$
$Q := a_{-1} \varphi_0$
$G := \psi_{-1} b_{-1}$

Claim: This is a topological structure of dim $d = 1$.

$$\Omega^{ch}(\mathbb{A}^d) := \Omega^{\otimes d}$$

"physics catch word" – this is VOA of $\beta\gamma$ bc system in d dimensions.

Lagrangian formulation

$$\gamma : \Sigma \to M$$

$$\omega \in \Omega^1(\Sigma, \gamma^* TM)$$

$$\omega = \beta + \bar\beta, \quad \beta \in \Omega^1_{\mathbb{C}} \quad (\bar\beta \in \bar\Omega^1, \text{ ignore}).$$

$$\text{Action } S(\omega, \gamma) = \int_\Sigma \langle \omega, d\gamma \rangle$$

$$= \int_\Sigma \langle \beta, \bar\partial \gamma \rangle + \langle \bar\beta, \partial\gamma \rangle$$

$$\Rightarrow \text{vertex alg } \beta_i(z)\gamma^j(w) = \frac{\delta_i^j}{z-w} + \text{reg}.$$

$$\Big(M \to \hat{M} = (M, \Omega^\circ M) = \Pi TM$$
$$= \{ \underbrace{\mathbb{R}^{0|1} \to M}_{\text{odd loops}} \} \Big).$$

<u>Facts</u>. There is a homomorphism

$$\mathrm{Diff}(\mathbb{A}^d) \longrightarrow \mathrm{Aut}(\hat\Omega^{ch}(\mathbb{A}^d))$$

switch to series in Fock repns.

γ	$b \leftrightarrow x$ on $(M = \mathbb{A}^d)$
c	$\varphi \leftrightarrow dx = \xi$
b	$\psi \leftrightarrow \partial_\xi$
β	$a \leftrightarrow \partial_x$

$$\mathcal{D}iff(\mathbb{A}^d) \longrightarrow Aut(\hat{\Omega}^{ch}(\mathbb{A}^d))$$

$\tilde{x} = g(x)$, $g \in \mathcal{D}iff$

$\tilde{\varphi} = g'$

$\tilde{\psi}$

$\tilde{a}$ – most tricky

same OPE, and in fact homomorphism.

$\Rightarrow$ can <u>localize</u>.

Structure fields

$\tilde{L} = L$

$\tilde{G} = G$

but $\boxed{\begin{aligned}\tilde{Q} &= Q\\ \tilde{J} &= J\end{aligned}}$ + anomaly $\sim \left[\operatorname{Tr}\log\left(\frac{\partial g^i}{\partial x^j}\right)(z)\right]'$
+ anomaly

represents $c_1(TM)$.

So, if M is CY, $c_1 = 0$.

$N = 2$ top. str.

But even in general, J_0, Q_0 survive.

Fermionic charge

differential $Q_0^2 = 0$
(Chiral de Rham differential)

$\Omega^{ch}(M)$ – sheaf of vertex algebras
with $d^{ch} = Q_0$.

not quasi coherent.

but has a filtration whose Gr is quasicoherent.

$\rightarrow \mathrm{Gr}\,\Omega^{ch} \simeq \mathcal{E}ll$

$H^\bullet(\Omega^{ch}(M))$ is VOA