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Formal group laws for infinite-dimensional Lie algebras

We want good integral forms for universal enveloping algebras (UEA) of Lie algebras

<u>Example</u> sl_2 E, F, H.

U = UEA generated (over $\mathbb{Q}$) by E, F, H.

Wrong answer: algebra generated by E, F, H over $\mathbb{Z}$.

<u>Correct form</u> (Kostant):

Take algebra generated by $\frac{E^n}{n!}$, $\frac{F^n}{n!}$

Example: we can construct $\exp(xE) = \sum \frac{x^n E^n}{n!}$

If U is a UEA of Lie algebra G

we have a <u>coproduct</u> map $\Delta : U \to U \otimes U$

$$\Delta(g) = g \otimes 1 + 1 \otimes g, \quad g \in G.$$

A <u>structural basis</u> for U is a basis Z_α

$\alpha \in \mathbb{Z}^{(i)}_{\geq 0}$.

seq of all nonnegative integers indexed by i

$$\boxed{\Delta Z_\alpha = \sum_{0 \leq \beta \leq \alpha} Z_\beta \otimes Z_{\alpha - \beta}}$$

Example: Lie algebra generated by D.

$$\text{U.E.A} \qquad 1,\ D,\ D^2/2!,\ D^3/3!,\ \cdots$$

$$D^{(0)},\ D^{(1)},\ D^{(2)},\ \cdots$$

$$\Delta D^{(1)} = 1 \otimes D^{(1)} + D^{(1)} \otimes 1$$

$$\Delta D^{(2)} = 1 \otimes D^{(2)} + D^{(1)} \otimes D^{(1)} + D^{(2)} \otimes 1.$$

<u>Meaning of structural basis</u>:

U - UEA

dual U^* - commutative algebra - (Completion of) coordinate ring.

U has structural basis

$\Longrightarrow$ U^* formal power series ring.

"U is smooth"

$U \otimes U \longrightarrow U$ (product)

$$\left.\begin{array}{c} U^* \longrightarrow \underbrace{U^* \hat{\otimes}\, U^*} \\ \uparrow \qquad\qquad\qquad\qquad \end{array}\right\}$$

formal power series ring in n-variables

formal power series ring in $2n$ variables

n formal power series in $2n$ variables.

So, Structural basis $\Longleftrightarrow$ Formal group law over $\mathbb{Z}$.

Finite dim semisimple Lie algebras : e_i, f_i, h_i

Kostant's form $e_i^n/n!$, $f_i^n/n!$

Same works for K-M algebras

e_i, f_i locally nilpotent

<u>Problem</u>: generalized KM algebras }
Virasoro algebra

$[L_i, L_j] = (i-j) L_{i+j} (+*)$ $\Rightarrow$ not enough locally nilpotent elements

If U has a structural basis, easy to check that all primitive elements of U are <u>liftable</u>

$\Delta(g) = g \otimes 1 + 1 \otimes g$

$\exists$ elements $g_0, g_1, g_2, \cdots$ $(g_0 = 1, g_1 = g)$ such that $\sum_{i \geq 0} g_i x^i$ is <u>grouplike</u>

$\Delta(a) = a \otimes a$

namely, can find $g_1, g_2, \cdots$

$\Delta g_1 = g_1 \otimes 1 + 1 \otimes g_1$

$\Delta g_2 = g_2 \otimes 1 + g_1 \otimes g_1 + 1 \otimes g_2$

Conversely, all primitive elements liftable
+ mild condition

$\Rightarrow$ structural basis exists

$\left(e^{(i)} = e^i / i! \quad \sum e^{(i)} x^i \text{ is grouplike} \right)$

If a, b liftable, so is $\lambda a + \mu b$ for $\lambda, \mu \in \mathbb{Z}$ (trivial to check)

If a, b liftable, so is $[a, b]$

(So we just have to check a set of generators is liftable)

Look at universal example:

U = non-commutative algebra generated by a_i, b_i, $i \geq 1$.

Coalgebra: $\Delta a_n = \sum a_i \otimes a_{n-i}$

Δb_n similarly

Can we find c_i in U, $c_1 = [a, b]$

$$\Delta c_n = \sum c_i \otimes c_{n-i} \ ?$$

Theorem: Suppose H Hopf algebra

$$H = \bigoplus_{n \geq 0} H_n, \quad H_0 = \mathbb{Z}$$

H_i finite, free over $\mathbb{Z}$.

If $V_p : H/pH \to H/pH$ is onto for all p, all primitive elements of H liftable.

If H is co-commutative Hopf algebra over $\mathbb{F}_p$ there is a homomorphism $V_p : H \to H$ (called shift or Verschiebung) dual to frobenius map $H^* \longrightarrow H^*$

$$x \longmapsto x^p$$

$\Delta a_n = \sum a_i \otimes a_{n-i}$, then $V_p(a_n) = 0$ if $p \nmid n$
a_n/p if $p \mid n$

So all a_i in image of V_p.

So c_n as above exist.

Open problem: find explicit c_n's.

Conclusion: to find a good integral form for UEA of a Lie algebra G, main step is to find liftings of a set of generators of ~~$\mathfrak{g}$~~ G.

(Example: can be used to find good UEA for some GKM, such as fake monster Lie algebra).

Example: Witt algebra: $[L_m, L_n] = (n-m) L_{n+m}$.

Problem: find (good) liftings for all elements L_i.

$L_0 : \binom{L_0}{i}$

$L_1 : L_1^n / n!$

$L_{-1} : L_{-1}^n / n!$

$L_2 : L_2^n / n!$ is wrong.

$1, L_2, \frac{L_2^2 + L_4}{2!}, \ldots$

- Construction of (good) liftings of L_m.

  Witt algebra = Lie algebra of derivations of $R = \mathbb{Z}[x][x^{-1}]$, $L_m = -x^{m+1}\frac{d}{dx}$.

  Look at $\overbrace{\text{automorphism}}^{a(\varepsilon)}$ of $R[[\varepsilon]]$ taking x to $x - \varepsilon x^{m+1}$

  $$a(\varepsilon) = 1 + L_m \varepsilon + O(\varepsilon^2).$$

  $\rightarrow$ $a(\varepsilon)$ lifts L_m in $(End(R) \otimes \mathbb{Q})[[\varepsilon]]$

  We want a lift in $U(Der(R) \otimes \mathbb{Q})[[\varepsilon]]$

  Look at $\exp(\log(a))$

  in $End(R) \otimes \mathbb{Q}$ $[[\varepsilon]]$

  becomes primitive, so in $Der(R) \otimes \mathbb{Q}[[\varepsilon]]$

  in $U(Der(R) \otimes \mathbb{Q})[[\varepsilon]]$.

  This gives liftings of all elements L_i in $U(Der(R) \otimes \mathbb{Q})[[\varepsilon]]$

  All coefficients of this liftings generate an integral form with structural basis.

  <u>Open problem</u> : Find <u>explicit</u> lifts of L_m.

Application to modula moonshine:

V = monster vertex algebra, over $\mathbb{Z}$.

Look at Tate cohomology

$$\underbrace{H^0(g,V) \oplus H^1(g,V)}$$ $g \in$ monster.

$H^0(g,V) = \dfrac{\text{fixed pts of } g)}{(1+g+\cdots+g^{p-1})V}$

$H^1(g,V) = \dfrac{\{v \in V \mid (1+g+\cdots+g^{p-1})v=0\}}{(1-g)V}$

Vertex superalgebra over F_p, $g^p=1$
acted on $Z_M(g)$.

$$Tr(h \mid H^*(g,V)) = Tr(gh \mid V).$$

Ryba: If $g^p=1$, hauptmodul of g has pos. coeffs
then there should be a vertex algebra V^p for $Z_M(g)$
with $Tr(h|V^p) = Tr(gh|V)$ over F_p.

Follows if $H^1(g,V)=0$.

Follows using integral form of Vir.