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Affine Kac-Moody algebras and geometric Langlands correspondance

$GL_n(\mathbb{Q}_p)$ $\qquad 0 \to \mathbb{C}K \to \hat{\mathfrak{g}} \to \mathfrak{g} \otimes \mathbb{C}((t)) \to 0$

$GL_n(\mathbb{F}_q((t)))$

- Local Langlands correspondance

  - Relate reps of $GL_n(K)$ to n-dimensional rep. of $Gal(\bar{K}/K)$ (Weil group)

Idea: Try to do something like this for representations of $\hat{\mathfrak{g}}$

- Global Langlands correspondance

F - global field, i.e. number field, or a function field (We will consider this case)

$F = \mathbb{F}_q(X)$, X - smooth proj curve $/\mathbb{F}_q$

$\bar{F}$ - closure of F. $\quad Gal(\bar{F}/F)$.

$$\left\{\begin{matrix}\text{irreducible} \\ n\text{-dim reps} \\ \text{of } Gal(\bar{F}/F)\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{cuspidal automorphic} \\ \text{reps of } GL_n(\mathbb{A})\end{matrix}\right\}.$$

(or automorphic functions)

$\mathbb{A}$:

some finite ext of ground field

$x \in X \rightsquigarrow$ completion of F, $K_x \simeq \mathbb{F}_{q_x}((t))$
$\cup$
$\mathcal{O}_x \simeq \mathbb{F}_{q_x}[[t]]$

$F \hookrightarrow \mathbb{A} = \prod'_{x \in X} K_x$ almost everywhere in $\mathcal{O}_x$

automorphic: roughly, can be realized in functions on $GL_n(F) \backslash GL_n(\mathbb{A})$

If Π is such, and $v \in \Pi$, $K \cdot v = v$
$\to$ Get a function on $GL_n(F) \backslash GL_n(\mathbb{A}) / K$.

$\to$ Lafforgue, for GL_n.

Replace GL_n by G - split reductive group over $\mathbb{F}_q$.

$\Rightarrow \quad Gal(\bar{F}/F) \to {}^L G \longleftrightarrow$ Reps of $G(\mathbb{A})$

Langlands dual gp.

$G \supset T$ lattice of char, cochar

$X^*(T) \qquad X_*(T)$
$\cup \qquad \cup$
$\Delta \qquad \Delta^\vee$

switch to get ${}^L G$

G	${}^L G$
GL_n	GL_n
SL_n	PGL_n
SO_{2n+1}	Sp_{2n}
SO_{2n}	SO_{2n}

$$G(\mathbb{A}) = \prod_{x \in X}{}' G(K_x) \quad \text{product of local factors}$$

$$\mathrm{Gal}(\bar{F}/F) \hookleftarrow \mathrm{Gal}(\bar{K}_x/K_x).$$

$\dashrightarrow$ close connection between local & global.

- Now let X be a curve $/ \mathbb{C}$.

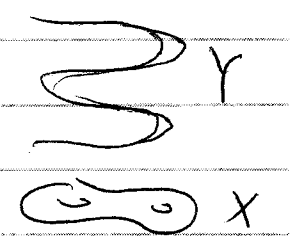
<u>LHS data</u>

for function field case, clear that

Galois gp $\rightsquigarrow$ fundamental gp $\pi_1(X)$.

So, homomorphisms $\mathrm{Gal}(\bar{F}/F) \to {}^L G$ become

$$\pi_1(X) \longrightarrow {}^L G(\mathbb{C})$$



extending fun field on X leads to covering spaces allowing ramification on pts

${}^L G$ – local system ; principal ${}^L G$-bdle on X + a connection.

<u>RHS</u> Unramified case $\to$ look at fns on (Hecke eigenfunctions)

$$G(F) \backslash G(\mathbb{A}) / G(\mathcal{O})$$

$K = G(\mathcal{O}) = \prod_{x \in X} G(\mathcal{O}_x)$

If it were set of $\mathbb{F}_q$-pts of a alg variety $/ \mathbb{F}_q$, get functions from sheaves.

Over $\mathbb{C}$ $\longrightarrow$ $\mathcal{D}$-modules (via Riemann Hilbert correspondance)

In fact this is set of isomorphism classes of prin. G-bundles on X, which is the set of points of algebraic moduli stack $= \boxed{Bun_G.}$

<u>So</u>: ${}^LG$-bdles on $X/\mathbb{C}$ with connection $\longleftrightarrow$ $\mathcal{D}$-modules on Bun_G.

Drinfeld Laumon Beilinson

(Must satisfy a Hecke condition)

General localization functor:

- Rep of Vir $\longrightarrow$ $\mathcal{D}$ (twisted) $-$mod on moduli space of curves (fiber $-$ coinvariants)
- Rep of $\hat{\mathfrak{g}}$ $\longrightarrow$ $\mathcal{D}$ (twisted) $-$mod on Bun_G.

If this rep is L_k - integrable vacuum rep $k \in \mathbb{Z}_+$ $\rightarrow$ the "sheaf of vacua" of WZW model

<u>Reference</u>: Frenkel, Ben-Zvi Book

"<u>Vertex algebras and algebraic Curves</u>"

To construct Hecke eigensheaves, we need reps of $\hat{\mathfrak{g}}$ of critical level, $k = -h^\vee$

Theorem (Feigin – F.) $\qquad U_{-h^\vee}(\hat{\mathfrak{g}}) = U(\hat{\mathfrak{g}})/(K + h^\vee)$

Let $\widetilde{U}_{-h^\vee}(\hat{\mathfrak{g}}) = \varprojlim_{N>0} U_{-h^\vee}(\hat{\mathfrak{g}}) \Big/ U_{-h^\vee}(\hat{\mathfrak{g}}) \cdot \mathfrak{g} \otimes t^N \mathbb{C}[[t]]$

This alg acts on "smooth reps"

i.e. such $\forall v$, $\exists N$, s.t. $\mathfrak{g} \otimes t^N \mathbb{C}[[t]]\, v = 0$

The center $Z(\hat{\mathfrak{g}})$ of $\widetilde{U}_{-h^\vee}(\hat{\mathfrak{g}})$ is canonically isomorphic to the alg $\mathrm{Fun}(\mathrm{Op}_{{}^L\mathfrak{g}}(D^\times))$.

$\mathrm{Op}_{{}^LG}(D^\times)$ – ${}^LG$ – opers on $D^\times$. $\left({}^LG \text{ – adjoint gp of } {}^L\mathfrak{g} \right)$

$\left(D^\times = \mathrm{Spec}\, \mathbb{C}((t)), \text{ punctured disc} \right)$

Def ${}^LG$ – oper on a curve X (or $D^\times$, or D) is a triple $(\mathcal{F}, \nabla, \mathcal{F}_{{}^LB})$.

$\mathcal{F}$: prin. ${}^LG$-bdle; ∇: connection on $\mathcal{F}$; $\mathcal{F}_{{}^LB}$: reduction to ${}^LB \subset {}^LG$

with <u>some condition</u>.

<u>Example</u> : ${}^L\mathfrak{g} = sl_2$, on $D^\times$

$\rightarrow$ Rk 2 vector bdle $\mathcal{F}$ on X , $\det \mathcal{F} = 0$.

Red to Borel $\rightarrow$ Rk 1 subbdle , $\mathcal{F}_{{}^LB}$.

Choose coord t.

The <u>condition</u> in this case is that one can bring ∇ to the form

$$\nabla = \partial_t + \begin{pmatrix} 0 & q(t) \\ 1 & 0 \end{pmatrix},$$

$$\sim \{ \partial_t^2 - q(t) \} \text{ projective connections}$$

$$\simeq Op_{sl_2}(D^\times)$$

$$\boxed{q(t) = \sum_{n > -M}^{\infty} q_n t^n \in \mathbb{C}((t)) = \sum_{m \in \mathbb{Z}} S_m t^{-m-2}}$$

$$Z(\hat{sl}_2) \simeq \tilde{\mathbb{C}}[q_n]_{n \in \mathbb{Z}} \quad \text{completion in negative direction}$$

$$\mathbb{C}[S_m] \xleftarrow{S_n \leftrightarrow q_{-2-n}}$$

$\rightarrow$ Center is gen. by Sugawara ops and transf as proj conn.

Let $V(\hat{\mathfrak{g}}) = \mathrm{Ind}_{\mathfrak{g}[[t]] \oplus \mathbb{C}K}^{\hat{\mathfrak{g}}} \mathbb{C}_{-h^\vee}$

have map $Z(\hat{\mathfrak{g}}) \longrightarrow \mathrm{End}_{\hat{\mathfrak{g}}} V(\hat{\mathfrak{g}}) =: \mathfrak{z}(\hat{\mathfrak{g}})$.

<u>Theorem (2)</u> This map is surjective and the image is $\mathrm{Fun}(Op_{{}^L\mathfrak{g}}(D))$.

(for sl_2 : $\mathrm{End} = \mathbb{C}[q_n]_{n \geq 0} = \mathbb{C}[S_m]_{m \leq -2}$

Given $\rho \in Op_{{}^L\mathfrak{g}}(D)$, construct a $\hat{\mathfrak{g}}$-module

$$V_\rho = V(\hat{\mathfrak{g}}) \Big/ \mathrm{Im}\left(\mathrm{Ker}\,\bar{\rho} : \mathfrak{z}(\hat{\mathfrak{g}}) \to \mathbb{C}\right)$$

<u>Theorem (3)</u> V_ρ is irreducible and $\neq 0$

These are all irred. unramified $\hat{\mathfrak{g}}_{-h^\vee}$-mod

$\left(\exists\, v \neq 0 \in V \text{ s.t. } \mathfrak{g}[[t]]v = 0 \right)$

Now by applying the localization functor Δ, get a twisted $\mathcal{D}$-module $\Delta(V_\rho)$ on Bun_G.

<u>Theorem</u> (Beilinson - Drinfeld)

(1) $\Delta(V_\rho) = 0$ if ρ cannot be extended to entire X

(2) Suppose ρ can be extended to $\tilde{\rho} \in Op_{{}^L\mathfrak{g}}(X)$, then $\Delta(V_\rho) \neq 0$ and it is a Hecke eigensheaf cor. to the ${}^LG$-local system $(\mathcal{F}, \nabla)$ (on Bun_G). So construct Langlands correspondence for ${}^LG$-loc. syst. admitting oper str

$\tilde{\rho} \in Op_{{}^L\mathfrak{g}}(X)$
$\quad \|$
$\{(\mathcal{F}, \nabla, \mathcal{F}_{{}^LB})\}$.

(3) This does not depend on the choice of x.

$Op_{{}^L\mathfrak{g}}(X) \hookrightarrow Loc.\,Syst_{{}^LG}(X)$ Here ${}^LG$ is of adjoint type

If can have $\mathcal{F}_{{}^LB}$, it is unique.