S. Ramanan   Introduction to Non-abelian Hodge theory

◉ General view

M - diff. manifold

de Rham: complex

$(*)$ $$A^{i-1}(M) \to A^i(M) \longrightarrow A^{i+1}(M)$$

de Rham: $H^i(M, (\mathbb{R} \text{ or } \mathbb{C}))$ & coh. of $(*)$ are isomorphic

M - complex manifold.

holomorphic de Rham complex of sheaves:

$(**)$ $$\overset{i-1}{\Lambda}(T^*) \longrightarrow \Lambda^i(T) \longrightarrow \Lambda^{i+1}(T)$$

~~deRh~~ hypercohomology of $(**)$ is iso to $H^i(M, \mathbb{C})$.

$(**)$ has a resolution:

$$0 \to \Lambda^i(T^*) \xrightarrow{d''} A^{i,0} \xrightarrow{d''} A^{i,1} \longrightarrow \cdots$$

soft resol'n. $\longrightarrow$ proof of the theorem.

This ~~resol'n~~ restatement is special because it involves the complex structure.

Hodge decomposition

Replace all differentials in $(**)$ by 0-map
(Trivial complex)

The complex reduces to

$$H^i = \sum_{p+q=i} H^q(M, \overset{p}{\Lambda}(T^*))$$

M - complex Kählerian $\Longrightarrow$ the hypercoho of $(**)$ and the trivial complex are isomorphic

Generalization:

Let $\mathbb{L}$ be a local system of $\mathbb{C}$-v.spaces.

All constructions before Hodge's thm go through.

$\rightsquigarrow$ $H^i(M,\mathbb{L}) \cong$ Hyper cohomology of the complex:

$$\mathbb{L}\otimes\Lambda^{i-1}(T^*) \longrightarrow \mathbb{L}\otimes\Lambda^i(T^*) \longrightarrow \cdots$$

or if denote $L = \mathbb{L}\otimes\mathbb{C}$

$$L\otimes\Lambda^{i-1}(T^*) \longrightarrow L\otimes\Lambda^i(T^*) \longrightarrow \cdots$$

Hodge theorem is not valid in general:

M - cpt. Riem surf. genus ≥ 2 $\quad \alpha$ - θ-charact. $\alpha^2 = K_X$

$$0 \to \alpha \to E \to \alpha^{-1} \to 0 \qquad H^1(\alpha^2) = H^1(K_X)$$

$\rightsquigarrow$ non-trivial extension

$\Rightarrow E$ comes from a local system. (Thm A. Weil

Terminology:

$$(\text{Local System}) \Leftrightarrow (\text{Repres of } \pi_1) \longleftrightarrow (\text{Vector bundles w/ flat connection})$$

Then we apply that construction to $\Lambda^i(T^*)\otimes_{\mathcal{O}} E \longrightarrow$ Hodge theorem is not true.

$\mathbb{H}^0$ of the triv exp $=$ $H^0(M,E) = H^0(M,\alpha)$ may be nontrivial.

$\mathbb{L}$-complex $\rightsquigarrow$ $H^0(M,\mathbb{L}) = 0$ since repres is irreducible

We will define a correspondence:

$\mathbb{L}$ - local system $\longmapsto$ Higgs pair (E,ω)

E - vector B.

$\omega \in \Gamma(T^* \otimes \mathrm{End}\, E)$ s.t. $\omega \wedge \omega = 0$

which induces a functor

(semi simple local system) $\longrightarrow$ (poly stable Higgs pairs of

$c_1(E) \cdot \varkappa^{n-1} = 0$ $\quad$ $\varkappa \in H^2(M)$ - the

$c_2(E) \cdot \varkappa^{n-2} = 0$ $\quad$ Kähler metric of M.

For a bundle we define $\deg E = c_1(E)\varkappa^{n-1}$ =: degree

$\mu(E) = c_2(E)\varkappa^{n-2} = \dfrac{\deg E}{rk(E)}$ =: slope

Def. $\quad$ E is stable if $\quad \forall\, F \subsetneq E \;\; \mu(F) < \mu(E)$

Similarly (E,ω) is stable if

$\forall\, F \subset E$, ω-invariant $\quad (\omega : E \to \Omega^1 \otimes E \Rightarrow F \longmapsto \Omega^1 \otimes F)$

$\Rightarrow \mu(F) < \mu(E)$

Polystable (E, ω) if $(E,\omega) = \bigoplus (E_i, \omega_i)$ $\quad \mu(E_i) = \mu(E_j)$

↑ stable

Def $\quad \omega$ - Higgs field.

We "modify" the trivial complex by the following procedure

(***) $\mathbb{L} \longrightarrow (E,\omega)$ Higgs $\longrightarrow$ complex

$E \otimes \Lambda^{i-1}(T^*) \xrightarrow{\wedge\omega} E \otimes \Lambda^i(T^*) \xrightarrow{\omega} \ldots$

Hodge theorem: The cohomology of (∗∗) and (∗) are iso,

Remark $(E,0)$ produces the trivial case

(Proof through Laplacians Δ, $\Delta_{\bar\partial}$ works for unitary local system)

Theorem (Narasimhan - Seshadri) (Unitary local systems) correspond to poly-stable bundles.

Tanaka yoga

Tensor structure on the category of polystable Higgs bundles:
given fixed slope

$(E,\omega), (E',\omega') \longmapsto ((E \otimes E'), \omega \otimes id + id \otimes \omega')$

The cat ~~forms~~ is abelian cat. (w/ addition)
duals: $(E,\omega)^* = (E^*, -{}^t\omega)$
trivial object: $(\mathcal{O},0)$
associative & commutative tensor product:

Tanaka category:
fix $m \in M$ (M - connected!)
$(E,\omega) \xrightarrow{F} (E_m) \in \underline{\text{Vect.sp}}$
fiber

correspondence which respects $+$, $\otimes$
In fact it is a faithful functor
The functor F $\underline{\text{poly - Higgs}} \longrightarrow \underline{\text{Vect}}$ makes (polystable pairs) Tanaka category
(fiber functor F)

Tanaka Fix group G, consider G-modules
reproduce G as a "dual" of (G-modules)

If G is reductive alg group $\Rightarrow G \longrightarrow \mathrm{Aut}^{\otimes}(F)$ is iso
↑ fiber functor

— Generally, for a Tanaka cat $\mathrm{Aut}^{\otimes}(F)$ is called Tanaka dual grp

Iso of Tanaka categories:

(s.s. Loc systems) $\longleftrightarrow$ (Poly stable Higgs bndls)
$\updownarrow$
this is a purely alg. object.

(E, ω), $\lambda \in \mathbb{C}^*$
$\mathbb{C}^*$ acts on (P.st Higgs)
$(E,\omega) \longmapsto (E, \lambda\omega)$

$\Rightarrow$ $\mathbb{C}^*$ acts on the Tanaka dual (to (Poly st. Higgs b)) $\longrightarrow \varpi_1$

Translation from $\pi_1 \longrightarrow \varpi_1$ - pro-alg. completion of π_1

Simpson: $SL(3,\mathbb{Z})$ is not the fund. grp of any Kähler manifold.

Can think of the $\mathbb{C}^*$ action as a ~~Hodge analogue~~.
non-abelian analogue to the Hodge decomposition.

Integrable system

M – curve

E - stable , consider (E, ω) is stable for any ω $H^0(K \otimes \mathrm{End}\, E) = T^*_E$ (Mod sp of vect. sp on M)

Consider map (Hitchin map)

$$H^0(K \otimes \mathrm{End}\, E) \longrightarrow \sum_{i=0}^{n} H^0(K^i) \quad \text{– affine space (in this case vector space)}$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ // ($^L G$ -opers)

$$T^*_E \longrightarrow \sum H^0(K^i)$$

gives a completely integrable system.

∃ Canonical square root of the canonical bundle on (Sl_n)

(Geometric Langlands program)