L. Lafforgue – Pavings of polyhedra, gluings of Schubert cells & compactifications of configuration spaces I

$E = E_0 \oplus \dots \oplus E_n \qquad E_I = \bigoplus_{\alpha \in I} E_\alpha \quad \forall I \subseteq \{0, \dots, n\}$

$Gr^{r,E} = \{ F \hookrightarrow E \ \ \dim F = r \}$

$\Rightarrow$ locally closed strata (thin Schubert cells)

$Gr_S^{r,E} = \{ F \hookrightarrow E, \ \dim(F \cap E_I) = d_I^S \ \forall I \}$

$(d_I^S)_{I \subseteq \{0 \dots n\}}$ satisfying: $\bullet\ d_\emptyset^S = 0, \ d^S_{\{0,\dots,n\}} = r$

$\bullet\ d_I^S + d_J^S \leq d_{I \cap J}^S + d_{I \cup J}^S \quad \forall I, J \qquad \bullet\ r_\alpha = r - d^S_{\{0,\dots,n\} \setminus \{\alpha\}} \leq \mathrm{rk}\, E_\alpha \quad 0 \leq \alpha \leq n$

$\dashrightarrow (d_I^S)$ = "matroid" of rank r over $\{0, \dots, n\}$

$Gr_S^{r,E} \circlearrowleft \mathrm{Aut}\, E_0 \times \dots \times \mathrm{Aut}\, E_n = \mathrm{Aut}\, E_\bullet \supset \mathbb{G}_m^{n+1}$

$\overline{Gr}_S^{r,E} = Gr_S^{r,E} / \mathbb{G}_m^{n+1}$

General problem – find (~~equivariant~~) compactifications of general quotients $\overline{Gr}_S^{r,E}$

1st case: $E_\alpha = \mathbb{A}^r \ \forall \alpha \qquad d_I^S = 0 \quad \forall I \subsetneq \{0, \dots, n\}$

Lemma $\quad Gr_S^{r,E} = GL_r^{n+1} / GL_r \qquad \overline{Gr}_S^{r,E} = PGL_r^{n+1} / PGL_r$

2nd case $\quad r_\alpha = 1 \ \forall \alpha \qquad \mathrm{rk}\, E_\alpha = 1$

$\Rightarrow$ configuration space of the matroid $S = (d_I^S)_I$

$C_S^{r,n} = \left\{ P_0, P_1, \dots P_n \in \mathbb{P}^{r-1} : \dim \underbrace{\{P_\alpha\}_{\alpha \in I}}_{\text{projective span}} = r - d^S_{\{0,\dots,n\} \setminus I} - 1 \right\}$

$\circlearrowleft$ PGL_r $\qquad C_S^{r,n} / PGL_r =: \overline{C}_S^{r,n}$

Theorem (Gelfand–MacPherson) The map $\quad (r_\alpha = 1 \text{ all } \alpha)$

$(F \subset E_0 \oplus \dots \oplus E_n) \longmapsto \{ \mathrm{Ker}(F \to E_\alpha) \in \mathbb{P}(F^\vee) \}$

induces an isomorphism $\overline{Gr}_S^{r,E} \xrightarrow{\sim} \overline{C}_S^{r,n}$

Remark $\quad r = 2$: classifying $n+1$ points on $\mathbb{P}^1 \Rightarrow \mathcal{M}_{0,n+1}$

$r = 3$: points in $\mathbb{P}^2$ with specified linear relations

Thales: Any scheme, integral of finite type over $\mathbb{Z}$ contains an open $\simeq C_S^{3,n}$

[Thales: can model addition & multiplication by straight line conditions on the plane ..]

Mnev 1986: $\overline{C}_S^{3,n}$ have arbitrary singularities & topology!
... X arbitrary affine finite type $/\mathbb{Z}$
$\Rightarrow$ integer N & a $C_S^{3,n} \simeq U \xhookrightarrow{\text{open}} X \times \mathbb{A}^N \to X$ affine

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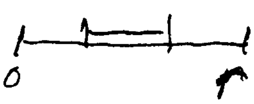
Consider Plücker embedding $Gr^{r,E} \hookrightarrow \mathbb{P}(\Lambda^r E)$

$$\Lambda^r E = \bigoplus_{\underline{i} \in S^{r,n}} \Lambda^{\underline{i}} E_\bullet \qquad S^{r,n} = \{ \underline{i} = (i_0 \dots i_n) \in \mathbb{N}^{n+1} / \textstyle\sum_j i_j = r \}$$

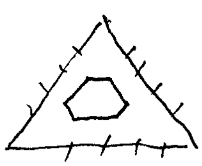
$$\Lambda^{\underline{i}} E_\bullet = \Lambda^{i_0} E_0 \otimes \dots \otimes \Lambda^{i_n} E_n$$

Consider an arbitrary matroid $(d_I^S)_I \Rightarrow$ convex polytope

$$S_{\mathbb{R}} = \{ (i_0, \dots, i_n) \in \mathbb{R}_+^{n+1} / i_0 + \dots + i_n = r, \ \sum_{\alpha \in I} i_\alpha \geq d_I^S \ \forall I \}$$
$$\cup$$
$$S = S_{\mathbb{R}} \cap S^{r,n} = S_{\mathbb{R}} \cap \mathbb{N}^{n+1} \quad \text{integer points}$$

$n=1$ 

$S^{r,n} \Rightarrow$ integer subinterval

$n=2$ 
$S^{r,2}$ sides parallel to triangle
– not true for higher n, faces can be much more complicated

Proposition In $Gr^{r,E} \hookrightarrow \mathbb{P}(\Lambda^r E) = \{ (X_{\underline{i}})_{\underline{i} \in S^{r,n}} \in \mathbb{G}_m \setminus \prod_{\underline{i}} \Lambda^{\underline{i}}_E \}$
the thin Schubert cell $Gr_S^{r,E}$ is defined by

- $X_{\underline{i}} = 0 \quad \forall \underline{i} \notin S$
- $X_{\underline{i}} \neq 0 \quad \forall \underline{i} \in S$

Proposition $S \in S^{r,n}$, $S_{\mathbb{R}}$ associated to a matroid $(d_I^S)_I$

i) $\forall I \quad d_I = \min \{ \sum_{\alpha \in I} i_\alpha / \underline{i} \in S \}$

$\Rightarrow$ can recover matroid from S

ii) $S_{\mathbb{R}}$ is generated by S integral points

iii) The faces of $S_{\mathbb{R}}$ are also matroid polytopes

iv) A convex polytope which admits a paving by matroid polytopes is a matroid polytope

v) $\dim S_{\mathbb{R}} = n-p \Rightarrow \exists!$ decomposition $\{0,..n\} = J_0 \amalg \ldots \amalg J_p$
$|J_i| = n_i + 1$, $r = r_0 + \cdots + r_p$
s.t. $S_{\mathbb{R}} = S_{\mathbb{R}}^0 \times \ldots \times S_{\mathbb{R}}^p$, $S = S^0 \times \ldots \times S^p$
each S^i a matroid polytope of maximal dimension in $S^{r_i, n_i} = \{ (i_\alpha)_{\alpha \in J_i} \in \mathbb{N}^{J_i} / \sum_{\alpha \in J_i} i_\alpha = r_i \}$

vi) Any matroid polytope $S \subseteq S^{r,n}$ of maximal dimension n contains a generating subfamily ($n+1$ pts generating integral lattice)

Lemma S = matroid polytope $\subseteq S^{r,n}$, S' = face of S

$$\mathbb{G}_m \backslash \prod_{\underline{i} \in S} (\Lambda^{\underline{i}} E_\bullet - \{0\}) \longrightarrow \mathbb{G}_m \backslash \prod_{\underline{i} \in S'} (\Lambda^{\underline{i}} E_\bullet - \{0\})$$
$$(X_{\underline{i}})_{\underline{i} \in S} \longmapsto (X_{\underline{i}})_{\underline{i} \in S'}$$

induces a morphism $Gr_S^{r,E} \longrightarrow Gr_{S'}^{r,E}$

eg S' defined in S by equation $\sum_{\alpha \in I} i_\alpha = d_I^S$

$$\Rightarrow (F \hookrightarrow E) \longmapsto (F \cap E_I) \oplus (F / F \cap E_I)$$
$$E_I \oplus E / E_I = E_I \oplus E_J = E.$$

Def Let $\underline{S}$ be a paving of $S^{r,n}$ by matroid polytopes s.t. for any cells S', S'' $S' \cap S''$ is a face of both S', S''.
Introduce closed subscheme $\boxed{Gr_{\underline{S}}^{r,E}} \hookrightarrow \mathbb{G}_m \backslash (\prod_{\underline{i} \in S} \Lambda^{\underline{i}} E_\bullet - 0)$
of families $(X_{\underline{i}})_{\underline{i} \in S}$ s.t. for any cell S' of $\underline{S}$ subfamily $(X_{\underline{i}})_{\underline{i} \in S'}$ defines a point in thin Schubert cell for S', $Gr_{S'}^{r,E}$

all coords nonzero

ie we're considering all the Schubert cells for faces of the pavings $Gr^{r,E}_{S'}$ as projective system

& take limit $Gr^{r,E}_{\underline{S}} = \varprojlim Gr^{r,E}_{S'}$

<u>Corollary</u> $Gr^{r,E}_{\underline{S}}$ classifies families $(F_{S'} \hookrightarrow E)_{S' \text{ cell of } \underline{S}}$

s.t. • $\forall S'$, $F_{S'} \in Gr^{r,E}_{S'}$

• S', S'' two cells with joint face of codim 1 defined by a single equation $\sum_{\alpha \in I} i_\alpha = d^{S'}_I = r - d^{S''}_J$ $J = \{0..n\} \setminus I$

$\Rightarrow$ gluing condition

$$0 \to F_{S'} \cap E_I \to F_{S'} \to \_ \to 0$$
$$0 \to F_{S''} \cap E_J \to F_{S''} \to \_ \to 0$$

<u>The toric variety of pavings of S</u>

$\underline{S}$ = paving of S by matroid polytopes

$\mathcal{C}^S_{\underline{S}} \subset \mathbb{R}^S$ cone of functions $v: S \to \mathbb{R}$ s.t. for any cell S' of $\underline{S}$, $\exists$ affine function $\ell: S \to \mathbb{R}$ verifying • $\ell \le v$ • $S' = \{ i \in S / \ell(i) = v(i) \}$

If $\mathcal{C}^S_{\underline{S}} \neq 0$ $\underline{S}$ is a <u>regular paving</u>

ϕ = trivial paving $= (S)$ $\Rightarrow \mathcal{C}^S_\phi = \{$ affine fns $\ell: S \to \mathbb{R}\}$

<u>$\forall S, \ \mathcal{C}^S_{\underline{S}} + \mathcal{C}^S_\phi = \mathcal{C}^S_{\underline{S}}$</u>

<u>Prop</u> The cones $\mathcal{C}^S_{\underline{S}} / \mathcal{C}^S_\phi \subset \mathbb{R}^S / \mathcal{C}^S_\phi$ is a fan

$\Rightarrow$ normal (quasiproj) toric variety $\mathcal{A}^S$ for torus

<u>$\mathcal{A}^S_\phi = \mathbb{G}_m^S / (\mathbb{G}_m^S)_\phi$</u> $\dashrightarrow$ affine fns $S \to \mathbb{G}_m$.

Regular pavings $\underline{S}$ of $S \longleftrightarrow$ orbits $\mathcal{A}^S_{\underline{S}}$ in $\mathcal{A}^S$.

$\mathcal{A}^S_{\underline{S}} \subset \overline{\mathcal{A}^S_{\underline{U}}}$ iff $\underline{S}$ is a refinement of $\underline{U}$.

$\mathcal{A}^S / \mathcal{A}^S_\phi$ toric stack : toric variety mod its torus, points $\longleftrightarrow$ pavings

<u>Theorem</u> In the scheme $\mathbb{A}^S \times \mathbb{G}_m \setminus \prod_{i \in S} (\Lambda^i E_\bullet \setminus \{0\})$

there exists a closed subscheme $\Omega^{S,E}$ s.t.

i) invariant by $\mathrm{Aut}\, E_\bullet$ & torus $\mathbb{G}_m^S / \mathbb{G}_m$
(on both factors : on $\mathbb{A}^S$ by inverse & on second factor componentwise)

ii) The fiber of $\Omega^{S,E} \longrightarrow \mathbb{A}^S$ over $1 \in \mathbb{A}^S_\emptyset \subset \mathbb{A}^S$ is
thin Schubert cell $\mathrm{Gr}_S^{r,E}$.

iii) The fiber of $\Omega^{S,E}$ over $\alpha_{\underline{S}} \in \mathbb{A}^S_{\underline{S}}$ distinguished point in $\underline{S}$ arbitr.
is $\mathrm{Gr}_{\underline{S}}^{r,E}$.

iv) The quotient $\overline{\Omega}^{S,E}$ of $\Omega^{S,E}$ by the free action of $\mathbb{G}_m^S / \mathbb{G}_m$
is a <u>projective</u> scheme endowed with a morphism
$\overline{\Omega}^{S,E} \longrightarrow \mathbb{A}^S / \mathbb{A}^S_\emptyset$

$\Rightarrow$ strata $\overline{\Omega}_{\underline{S}}^{S,E}$ in $\overline{\Omega}^{S,E}$. The open stratum
is $\overline{\Omega}_\emptyset^{S,E} = \mathrm{Gr}_S^{r,E} / (\mathbb{G}_m^S)_\emptyset / \mathbb{G}_m) = \overline{\mathrm{Gr}}_S^{r,E}$

<u>Working conjecture</u> $\forall S$ the open stratum
$\overline{\Omega}_\emptyset^{S,E} = \overline{\mathrm{Gr}}_S^{r,E}$ is scheme theoretically dense in $\overline{\Omega}^{S,E}$.