T. Haines - Weight filtration on nearby cycles for Shimura varieties (w. U. Görtz)

PEL type Shimura varieties $Sh(\underline{G}, h, K^p K_p)$

assume $K_p \subset \underline{G}(\mathbb{Q}_p)$ Iwahori subgroup

$h \to$ minuscule dominant coweight μ.

Assume $G = \underline{G}_{\mathbb{Q}_p} = GL_n$ or GSp_{2n} :

so $\mu = (1^r, 0^{n-r})$ (GL_n) or $(1^n, 0^n)$ (for $GSp(2n)$)

Local model $\mathcal{M}/\mathbb{Z}_p$ proper: $\mathcal{M}_{\mathbb{Q}_p} = \mathcal{G}r_{\mu, \mathbb{Q}_p} \subset \mathcal{G}r_{\mathbb{Q}_p}$ (generic fiber)

$\mathcal{G}r_{\mathbb{Q}_p} = G(\mathbb{Q}_p((t)))/G(\mathbb{Q}_p[[t]])$

special fiber $\mathcal{M}_{\mathbb{F}_p} \hookrightarrow \mathcal{F}l_{\mathbb{F}_p} = G(\mathbb{F}_p((t)))/I_w$ ---> Iwahori
- affine flag

[Kottwitz-Rapoport]

$\mathcal{M}_{\mathbb{F}_p}$ = Union of fin many Iwahori orbits, labelled by $\widetilde{W}_{aff}(G) \supset Adm(\mu) = \{ x \in \widetilde{W} \mid x \leq t_{w\mu}, w \in W\}$

-- closed under Bruhat order.

eg $G = GL_n$ $\mu = (1^r, 0^{n-r})$ ($\underline{G}$ "fake unitary group"

E imaginary quadratic where p splits.
$\mathbb{Q}$ $D, *$ n^2 dim division algebra / E with involution

$\underline{G} = D^\times$: $D \otimes \mathbb{C} \cong M_n(\mathbb{C})$

$h : \mathbb{C}^* \to$ diagonal matrix $a+ib \mapsto ((a-ib)^r, (a+ib)^{n-r})$

K_p = Iwahori) (note : $\underline{G}(\mathbb{R}) \cong$ " $U(r, n-r)$)

R a $\mathbb{Z}_p$-algebra, $M(R)$ = isom classes of periodic lattice chains $\{ \mathcal{L}_0 = I_0 \subset \mathcal{L}_1 \subset \dots \subset \mathcal{L}_n = (t+p)^{-1} \mathcal{L}_0 \}$

$R[t]$-submodules of $R[t, t^{-1}, (t+p)^{-1}]^n$.

If $V_i = (t+p)^{-1} R[t]^i \oplus R[t]^{n-i}$

Fix relative position wrt V_i :

1. $t V_i \subset \mathcal{L}_i \subset V_i$
2. $V_i / \mathcal{L}_i$ is R-locally a free factor of $V_i / t V_i$, of rank r.

"Extended" version M_μ : μ dominant coweight, not nec. minuscule. $\mathcal{L}_\bullet = (\mathcal{L}_0 \subset \mathcal{L}_1 \subset \dots \subset (t+p)^{-1} \mathcal{L}_0)$ s.t.

1' $\mathcal{L}_i \xrightarrow{\leq \mu} V_i$ V_i : relative position wrt to

Thus have degeneration of piece of affine grassmanian to piece of affine flags:

$$\mathcal{F}\ell \supset M_{\mathbb{F}_p} \hookleftarrow\!\!\longrightarrow M_\mu \hookleftarrow\!\!\longrightarrow M_{\mathbb{Q}_p} = \overline{\mathcal{G}r_\mu} \subset \mathcal{G}r_{\mathbb{Q}_p}$$

Consider IC complex of $\mathcal{G}r_\mu$, $IC'_\mu = j_{\mu !*} \mathbb{Q}_\ell [\ell(\mu)]$

$\Rightarrow R\Psi = R\Psi(IC'_\mu) \in Perv_{I_w}(\mathcal{F}\ell)$

"Kottwitz conjecture"

<u>Theorem</u> (Haines-Ngô, Gaitsgory) $Tr(Fr_q, R\Psi) = q_\mu^{\frac{1}{2}} \varepsilon_\mu \sum_{\lambda \leq \mu} m_\mu(\lambda) Z_\lambda$

(apply fonctions-faisceaux to get element of Iwahori-Hecke algebra

here $q_w = q^{\ell(w)}$, $\varepsilon_w = (-1)^{\ell(w)}$, Z_λ = Bernstein functions — give basis for center $Z(\mathcal{H}_{I_w})$

$Z_\lambda = \sum_{\nu \in W(\lambda)} \Theta_\nu$ Θ_ν classes of line bundles

- prove by showing $R\Psi$ is a <u>central sheaf</u> wrt convolution then characterize its Satake transform (after pushforward to affine Grassmannian).

Since $Tr(Fr_q, R\Psi) \in \mathbb{Z}[q]$ $\Rightarrow$ gradeds of weight filtration (semisimplified) are $\bigoplus_j (gr_j^W R\Psi)^{ss} = \bigoplus_{w \in Adm} \bigoplus_{i=0}^{\ell(\mu)-\ell(w)} IC'_w(-i)^{\oplus m(w,i)}$

??

Algorithm to compute multiplicities $m(w,i)$:

$$\sum_i m(x,i) q^i = \underbrace{Tr(Fr_q, R\Psi_x)}_{\text{known by explicit formula}} - \underbrace{\sum_{\substack{w \in Adm \\ x < w}} \varepsilon_w m(w,i) q^i P_{x,w}}_{\text{known by induction on Bruhat order}}$$

<u>Conjecture</u> multiplicities of most singular point

$m(\tau, i) = \dim IH^{2i}(M_{\mathbb{Q}_p})$ Betti #s of generic fibers

Q: are the other multiplicities "cohomological"?

<u>Theorem</u> (Görtz-Haines) $\forall w$ $\sum_i m(w,i) q^i = \varepsilon_\mu Tr(Fr_q, H^0(M_{\bar{\mathbb{F}}_q}, R\Psi \overset{L}{\otimes} I(\mathcal{B}^w)))$

- new sheaf $R\Psi \overset{L}{\otimes} IC(\mathcal{B}^w)$ — finite <u>codimension</u> piece of affine flags where $\mathcal{B} = Iw \subset G(k((t)))$

$\mathcal{N}^- = \left(G(k[t^{-1}]) \xrightarrow{t^{-1} \mapsto 0} G \right)^{-1} (N^-)$

$\mathcal{B}$ orbits on $\mathcal{F}\ell \leftrightarrow \mathcal{N}^-$ orbits $\leftrightarrow \widetilde{W}_{aff}(G)$

$\mathcal{B}^w$: $\mathcal{N}^-$-orbit, codimension = length (w). $\mathcal{B}_w$ = Iw orbits

$\mathcal{B}^{\bar{w}} = \coprod_{y \geq w} \mathcal{B}^y$

What is $IC(\mathcal{B}^{\bar{w}})$? ∞-dim orbit etc.... Use construction of Faltings... Work in finite dimensional quotients:

Choose $\tilde{w}$ large s.t. $Adm(\mu) \subset \{x \leq \tilde{w}\}$

$\Omega = \bigcup_{w \leq \tilde{w}} w \mathcal{N}^- x = \coprod_{y \leq \tilde{w}} \mathcal{B}^y$: open, $\mathcal{N}^-$-invariant, contains $\mathcal{B}_y$ $\forall y \leq \tilde{w}$.

Choose $n \gg 0$ s.t. $\mathcal{N}^-(n) \subset \mathcal{N}^- \cap w \mathcal{N}^- w^{-1}$ $\forall w \leq \tilde{w}$

$\mathcal{N}^-(n) = \mathcal{N}^- \cap \left(G(k[t^{-1}]) \to G(k[t^{-1}]/t^{-n}) \right)^{-1} (T(k[t^{-1}]/t^{-n}))$

Then there is a quotient (covered by affine spaces) $\Omega \xrightarrow{\pi} \mathcal{N}^-(n) \backslash \Omega$ --- smooth

$$\begin{array}{ccc} \Omega & \xrightarrow{\pi} & \mathcal{N}^-(n)\backslash\Omega \\ \cup & & \cup \\ \mathcal{B}^{\bar{w}} \cap \Omega & \longrightarrow & \pi(\mathcal{B}^{\bar{w}} \cap \Omega) \end{array}$$

<u>Def</u> $IC(\mathcal{B}^{\bar{w}}) = \pi^*(IC(\pi(\mathcal{B}^{\bar{w}} \cap \Omega)))$

<u>Properties</u> (1) $IC(\pi(\mathcal{B}^{\bar{w}} \cap \Omega))|_{\pi(\mathcal{B}_y^{\bar{w}})} = IC(\mathcal{B}_y^{\bar{w}})$ $\quad (\mathcal{B}_y^{\bar{w}} = \mathcal{B}_y \cap \mathcal{B}^{\bar{w}})$

(2) "$\mathcal{N}^-$-equivariance" : $\forall z' \in \mathcal{B}_y^z$ $(y \geq z \geq w)$.

we have $IC_{\pi(z)}(\pi(\mathcal{B}^{\bar{w}} \cap \Omega)) = IC_{\pi(z')}(-)$.

For this need $\mathcal{N}^-(n) \backslash \mathcal{N}^- \underset{pr_2}{\overset{m}{\rightrightarrows}} \pi(\mathcal{B}^{\bar{w}} \cap \Omega)$ are both smooth

Use Kazhdan-Lusztig strategy to get $Tr(Fr_q, IC_x(\mathcal{B}^{\bar{w}})) = Q_{w,x}(q)$

Q = inverse Kazhdan-Lusztig polynomials.

<u>Corollary</u> $w = \tau$ most singular point, $\mathcal{B}^{\bar{\tau}} = \mathcal{F}\ell$ $\quad IC(\mathcal{B}^{\bar{\tau}}) = \mathbb{Q}_\ell$

$f : \mathcal{F}\ell_S \longrightarrow gr_S$, $(\mathcal{L}_\cdot) \mapsto \mathcal{L}_0$

$f_* R\Psi = IC_{\mu,S}$ $\quad \Longrightarrow$ the conjecture

<u>Generalization</u> to $Perv_{Iw}(\mathcal{F}\ell)$, $d = \dim Supp\, \mathcal{F}$ & suppose

$\mathcal{F}(d/2)$ has filtration by $j_{w_1!}\,\mathbb{Q}_\ell[\ell(w_1)](\frac{\ell(w_1)}{2}) *$
$j_{w_2 *}\,\mathbb{Q}_\ell[\ell(w_2)](\ell(w_2)/2)$

s.t. $d \equiv \ell(w_1) - \ell(w_2) \pmod 2$

Then $\bigoplus (gr_i^W \mathcal{F})^{ss} = \bigoplus_{w \in \mathrm{Supp}\,\mathcal{F}} \bigoplus_{i \in I(w) + \mathbb{Z}_{\geq 0}} IC_w(-i)^{\oplus m(w,i)}$

and $\sum_i m(w,i) q^i = \sum_d q^d \,\mathrm{Tr}(Fr_q, H^\bullet(\mathcal{F}\ell, D\mathcal{F} \otimes^L IC(\overline{BwB})))$