Cohomology ring of crepant resolution of orbifolds

Orbifold. Topological space X + orbifold structure

orbifold str = { $p \in X$ has an orbifold chart U_p "uniformized" by (V_p, G_p, π_p) $\pi_p: V_p \to U_p = V_p/G_p$

finite, but does not have to act effectively

G_p – effective $\iff$ X – reduced

Ex: (1) $X = \cdot \circlearrowleft G$ – orbifold

(2) $X = \mathbb{C}^n/G$ $G \subset GL(n, \mathbb{C})$

(3) M – algebraic surface, or symplectic 4-mfd, symmetric group S^n acts on M^n, $X = M^n/S^n$ – symplectic orbifold

(4) "Global quotient": Y – smooth, G finite group acts on Y

$X = Y/G$

(5) "Non-global quotient" wei

Weighted proj space $WP(d_1, \dots d_n) = S^{2n-1}/S^1$

$(z_1, \dots z_n) \to (e^{2\pi i d_1 \alpha} z_1, \dots e^{2\pi i d_n \alpha} z_n)$

$H \subset WP(d_1, \dots d_n)$

Calabi–Yau orbifolds – appears in string theory

Crepant resolution:

X – complex, reduced, Gorenstein orbifold

$\Downarrow$

Canonical bundle K_X is well-defined

(in general K_X is an orbifold vector bundle)

$\pi : Y \to X$ – crepant resolution

$\Leftrightarrow$ (1) π is a resolution i.e. π is biholomorphic on a dense open set

(2) $\pi^* K_X = K_Y$

a 'minimality' condition

Known results: (1) If $\dim X \leq 3$, crepant resolutions always exist

(2) If $\dim X \geq 3$, crepant resolutions are not unique. Different ones are related by "K-equivalence"

$$\begin{tikzcd} & Z \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] & \\ X \arrow[rr, no head, squiggly, "K\text{-equiv}"] & & Y \end{tikzcd} \qquad \text{s.t. } \varphi^* K_X = \psi^* K_Y$$

(3) $\dim X \geq 4$, crepant resolution may not exist, but there are many very interesting example.

(4) Most famous example

$M^{[n]}$ = Hilbert scheme of zero-dimension subscheme of length n

Extensive work.

(1) Go Voisin: M - symplectic mfd, $M^{[n]}$ – symplectic Hilbert scheme

Cohomology of Y

Extensive work on $M^{[n]}$ by many people. Göttsche, Nakajima — : Euler number, cohomology group K-theory, Motive.

Unknown: ring structure of $M^{[n]}$

Question 1: How to compute ring structure of $M^{(n)}$ and crepant resolution of Y in general.

Question: What kind of topological inv K-equiv ~~will~~ preserve?

~~General ph~~

The scheme of the talk: $H^*(Y)$ can be computed via orbifold cohomology ring of $H^*_{orb}(X)$ + quantum correction.

Answer to qu. 1 & 2: Two precise conjectures.

An short introduction to Orbifold cohomology ring (Chen-Ruan)

X - orbifold

Orbifold cohomology (Chen-Ruan)

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A cohomology theory of orbifold motivated by Orbifold string theory

classical cohomology + "stringy correction"

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Nontwisted sector + "Twisted sector"

Main ingredient

① Twisted sector

patching : $q \in U_q \subset U_p$ $\quad i: U_q \to U_p$ can be lift to

$i_*: V_q \to V_p$ $\quad i_\# \; G_q \rightarrowtail G_p$ unique up to conjugation

$g \in G_q$, $\quad (g)_{G_q} \sim (i_\#(g))_{G_p}$

$T_1 =$ set of equivalence class

$(g) \in T_1 \longrightarrow$ a sector $X_{(g)} = \{ (x, (g')_{G_x}), g' \in G_x, (g') \in (g) \}$

Twisted sector : $X_{(g)}$, $g \neq 1$

Nontwisted sector : $X_{(1)} = X$

$$\widetilde{\Sigma_1 X} = \bigsqcup_{(g) \in T_1} X_{(g)}$$

Ex: $X = Y/G$, $\quad \widetilde{\Sigma_1 X} = \bigsqcup_{(g)} Y_g / C(g)$

② Degree shifting: X^{2n} — almost complex orbifold

$x \in X_{(g)}$ $\quad g \curvearrowright T_x X$

diagonalize the action of g

$$g = \begin{pmatrix} e^{2\pi i \frac{m_1}{m}} & & \\ & \ddots & \\ & & e^{2\pi i \frac{m_n}{m}} \end{pmatrix} \qquad 0 \le m_i < m$$

Degree shifting number $\quad \iota_{(g)} = \sum \frac{m_i}{m} \in \mathbb{Q}$

$$H^*_{orb}(X, \mathbb{Q}) = \bigoplus_{(g) \in T_1} H^*(X_{(g)}, \mathbb{Q})[-2\iota_{(g)}]$$

③ Poincare Duality

$$H^p(X_{(g)}, \mathbb{Q})[-2\iota_{(g)}] \otimes H^{2n-p}(X_{(g^{-1})}, \mathbb{Q})[-2\iota_{(g^{-1})}] \to \mathbb{Q}$$

$$\langle \alpha, \beta \rangle_{orb} = \int_{X_{(g)}} \alpha \wedge I^* \beta$$

$$I: X_{(g)} \to X_{(g^{-1})}, \quad I(x, (g)_{G_x}) = (x, (g^{-1})_{G_x})$$

Key: $\quad \iota_{(g)} + \iota_{(g^{-1})} = \operatorname{codim}_{\mathbb{C}} X_{(g)}$

④ Cup product

$$\alpha \in H^p(X_{(g_1)}, \mathbb{Q})[-2\iota_{(g_1)}], \ \beta \in H^q(X_{(g_2)}, \mathbb{Q})[-2\iota_{(g_2)}]$$

$$\alpha \cup \beta \in H^{p+q}(X, \mathbb{Q}) = \bigoplus_{(g) \in T_1} H^{p+q}(X_{(g)}, \mathbb{Q})[-2\iota_{(g)}]$$

$$\alpha \cup \beta = \sum_{\substack{(h_1, h_2) \in T_2 \\ h_1 \in (g_1) \\ h_2 \in (g_2)}} (\alpha \cup \beta)_{(h_1 h_2)}$$

T_k = set of equivalence class of $(g_1 \cdots g_k)_{G_x}$

$$\langle (\alpha \cup \beta)_{(h_1 h_2)}, \gamma \rangle = \int_{X_{(h_1, h_2)}} e_1^* \alpha \wedge e_2^* \beta \wedge e_3^* \gamma \wedge e(E_{(h_1, h_2)})$$

$$X_{(h_1, h_2)} = \{ (x, (h_1', h_2')_{G_x}), \ (h_1', h_2')_{G_x} \in (h_1, h_2) \}$$

$e_1 : X_{(h_1, h_2)} \to X_{(h_1)} : (x, (h_1', h_2'))_{G_x} \to (x, (h_1'))_{G_x}$

$e_2 : X_{(h_1, h_2)} \to X_{(h_2)}$

$e_3 : X_{(h_1, h_2)} \to X_{((h_1 h_2)^{-1})} : (x, (h_1', h_2'))_{G_x} \to (x, (h_1' h$

Remarks: ① $H^*_{orb}(X, \mathbb{Q})$ is rationally graded in general

X - Gorenstein $\Rightarrow$ integrally graded

② $H^*_{orb}(X, \mathbb{Q})$ is much easier to be computed than that of crepant resolution Y.

Ex: M^n/S_n for any M : Göttsche - Fantechi - Urib

However, $H^*(M^{[n]}, \mathbb{Q})$ is unknown in general.

Ex: ① $X = \cdot /G$ $\qquad H^*_{orb}(X, \mathbb{Q}) = Z(\mathbb{Q}[G])$

center of group ring

② $X = \mathbb{C}^n/G \qquad G \subset GL(n, \mathbb{C})$

$$H^*_{orb}(X, \mathbb{Q}) \overset{\text{additive}}{\cong} Z(\mathbb{Q}[G])$$ with slightly different product

③ $X = WP(d_1, d_2)$

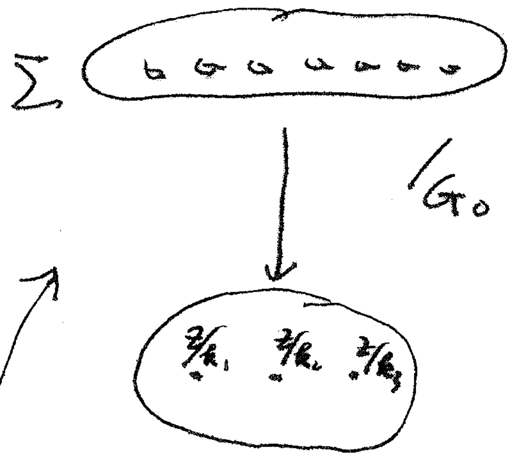
$$H^*_{orb}(X, \mathbb{Q}) = \mathbb{Q}[\alpha, \beta] / \{\alpha^{d_1} = \beta^{d_2},\ \alpha^{d_1+1} = \beta^{d_2+1} = 0\}$$

$\deg \alpha = \frac{2}{d_1} \qquad \deg \beta = \frac{2}{d_2}$

Construction of $E_{(h_1, h_2)} \longrightarrow X_{(h_1, h_2)}$

$G_0 = \langle h_1, h_2 \rangle$ — subgroup generated by h_1, h_2

$k_1 = \mathrm{ord}(h_1) \qquad k_2 = \mathrm{ord}(h_2) \qquad k_3 = \mathrm{ord}(h_1 h_2) = \mathrm{ord}((h_1 h_2)^{-1})$



$$\pi_1^{orb}(S^2) = \{ \lambda_1, \lambda_2, \lambda_3 \mid \lambda_1^{k_1} = 1 \quad \lambda_2^{k_2} = 1 \quad \lambda_3^{k_3} = 1 \quad \lambda_1 \lambda_2 \lambda_3 = 1 \}$$

$$\pi_1^{orb}(S^2) \longrightarrow G_0 \longrightarrow 1$$

$$E_{(h_1, h_2)} = \left(H^{0,1}(\Sigma) \otimes e^*_{(h_1, h_2)} TX \right)^{G_0}$$

$$\downarrow$$

$$X_{(h_1, h_2)}$$

Conjectures:

An easier case:

Cohomological Hyperkahler Resolution Conjecture (Ruan): Suppose $Y \to X$ is a hyperkahler resolution.

Then, $H^*(Y, \mathbb{Q})$, $H^*_{orb}(X, \mathbb{Q})$ are isomorphic as ring

Remarks: ① Conjecture is false without hyperkahler condition
② conjecture was solved by Lehn-Sorger-Fantech-Göttsche-Uribe for the case $X = M^{[n]}$, $M = T^4, K3$

③

General case:

Quantum corrected cohomology:

$\pi: Y \to X$, $A_1 \ldots A_k$ - integral basis of "exceptional rational curve"

i.e $\pi_*[A_i] = 0$

$$\langle \alpha, \beta, \gamma \rangle_{qc}(q_1 \ldots q_k) = \sum_{a_1 \ldots a_k} \Psi^X_{\sum a_i A_i}(\alpha, \beta, \gamma)\, q_1^{a_1} \ldots q_k^{a_k}$$

analytic function of quantum variable $q_1 \ldots q_k$

Set $q_i = -1$ $\quad \langle \alpha, \beta, \gamma \rangle_{qc} = \langle \alpha, \beta, \gamma \rangle_{qc}(-1, \ldots, -1)$

Define $\alpha *_{qc} \beta$ by equation $\langle \alpha *_{qc} \beta, \gamma \rangle = \langle \alpha, \beta, \gamma \rangle_{qc}$

Then, $\alpha \cup_\pi \beta = \alpha \cup \beta + \alpha *_{qc} \beta$

Cohomological Crepant Resolution Conj:

$H^*_\pi(Y, \mathbb{Q})$ is ring isomorphic to orbifold cohomology ring $H^*_{orb}(X, \mathbb{Q})$. (up to a sign)

$\pi : X \dashrightarrow X'$ birational map

X, X' – K – equivalent

Cohomological Minimal Model conjecture : $H^{*}_{\pi}(X, \mathbb{C})$ is ring isomorphic to $H^{*}_{\pi^{-1}}(X', \mathbb{C})$

Reasoning :

physical : shifting of value of B– field for conformal field theory after quantization (Wendland)

Mathematical : Morrison, An-Min Li – Ruan : $\pi : X^3 \dashrightarrow X'^3$ Flop

π induces an isomorphism on quantum cohomology after a change of quantum variable $q \to \frac{1}{q}$

If we set $q = \frac{1}{q} = \lambda \Rightarrow \lambda^2 = 1 \quad \lambda = \pm 1$

$\lambda = 1$ – pole no good

$\lambda = -1$

$\Uparrow$

LRL (~~June~~ An-Min Li – Ruan , Jun Li) – surgery formula

symplectic geometry algebraic geometry

Evidence for CCRC . Ex 1 . True for $M^{[2]}$ (Li - Qin)

Hope . $M^{[3]}$? (Thesis problem for a student)

Ex 2 . S – K3 with involution σ , E – elliptic curve with involution τ

$\widetilde{S \times E / \sigma \times \tau} \to S \times E / \sigma \times \tau$ (Borcea – Voisin 3-fold)

Evidence for CMMC Ex 1 $\pi : X^3 \to X'^3$ – smooth flop

Ex 2

$$\begin{array}{ccc} & X & \\ \nearrow & & \searrow \\ \mathbb{P}^2 \subset X^4 & \overset{\pi}{\dashrightarrow} & X'^4 \supset \mathbb{P}^2 \end{array}$$

$N_{\mathbb{P}^2} = T^*\mathbb{P}^2$ π – Mukai transform

Wanchuan Zhang . CMMC – true for 4-dim Mukai transform

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